Probabilistic Coherence Spaces: the Free Exponential Modality.

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joint work with:
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GDRI Linear Logic

Exponentials in Models of Linear Logic

Resource aware logic widely used to study semantics of computation.

- multiplicatives, additive connectors: $\otimes, \invamp, \&, \oplus$
- exponential structure: $!, ?$ $\Rightarrow$ in a model: several non isomorphic ! can coexist.

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  $\Rightarrow$ Finite model of λ-calculus.
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Alternative definition: by duality. $(|X|, \frown)$ as $(|X|, Cl(X))$

$a \perp b$ if $\text{card}(a \cap b) \leq 1$.
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Example (**Rel**: category of sets and relations.)

- Exponential multiset: $!A = \mathcal{M}_{\text{fin}}(A)$.
- Exponentials with infinite multiplicities [CarraroEhrhardSalibra'10]

Lafont's model: Free Commutative Comonoids

Category of commutative comonoids

comonoid: it is the given of: $\mathbf{1} \xleftarrow{w_C} C \xrightarrow{c_C} C \otimes C$

morphism of comonoids: respects the comonoid structure.

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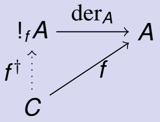
Universal Property

for every $(C, \mathrm{w}_C, \mathrm{c}_C)$ and $f : C \to A$, there exists a unique $f^\dagger$ morphism of comonoid with:



Lafont's Model

Theorem (Lafont)

A $\star$-autonomous category C is a model of linear logic if:

- *it has finite products and,*
- *for every object A, the free commutative comonoid generated by A exists.*

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Genericity of $!_f$: Decomposing $!A$ using $!_f A$.

$$\begin{array}{ccc} !_f A & \xrightarrow{\mathrm{der}_{fA}} & A \\ \exists ! \mathrm{der}^{\dagger} \uparrow & \nearrow_{\mathrm{der}} & \\ C = !A & & \end{array}$$

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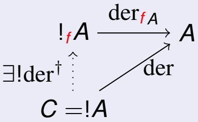

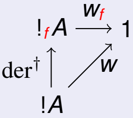

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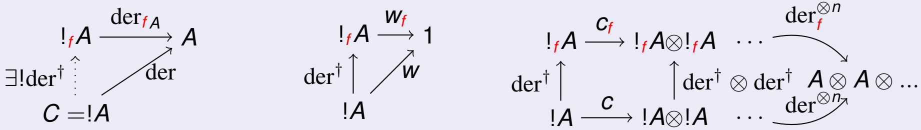

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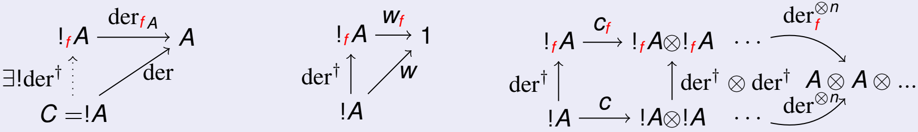

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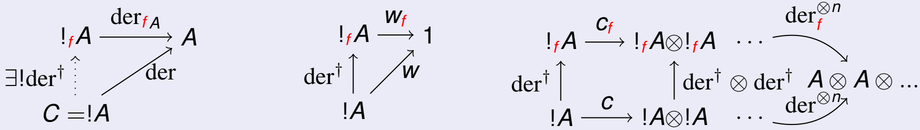

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Our Goal

Does the model of probabilistic coherence spaces (next slides) has a free exponential ?

Probabilistic coherence spaces (PCS)

- pre-PCS: $(|\mathcal{A}|, \mathrm{P}(\mathcal{A}))$, with $\mathrm{P}(\mathcal{A}) \subseteq (\mathbb{R}^+)^{|\mathcal{A}|}$.
- scalar product: $u, v \in (\mathbb{R}^+)^{|\mathcal{A}|}$:
  $\langle u, v\rangle \in (\mathbb{R}^+) \cup \{\infty\}$.
- Definition by duality:
  - for $u, v \in (\mathbb{R}^+)^{|\mathcal{A}|}$, $u \perp v$ iff $\langle u, v\rangle \leq 1$.
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Morphisms of PCS

- matrices $\phi \in \mathbb{R}^{+|\mathcal{A}| \times |\mathcal{B}|}$
  $=$ linear functions
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Unit object: 1

$|1| = \{\star\}$
$\mathrm{P}(1) = [0, 1]$

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$\Rightarrow$ sub-probability distributions on booleans.

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$\Rightarrow$ Markov transitions.

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Exponential comonad in Danos&Ehrhard 2011

- web: $|!_a\mathcal{A}| = \mathcal{M}_{\mathrm{f}}(|\mathcal{A}|)$
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functor on $\phi \in \mathrm{PCoh}(\mathcal{A},\mathcal{B})$	co-unit der $\mathrm{PCoh}(!_a\mathcal{A},\mathcal{A})$	co-multiplication digg $\mathrm{PCoh}(!_a\mathcal{A}, !_a!_a\mathcal{A})$
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Theorem

- **PCoh**$_{!a}$: *full abstraction for CBN probabilistic PCF. [EPT POPL'2014].*
- **PCoh**$^{!a}$: *full abstraction for a probabilistic version of Levy's CBPV [ET'2016].*

The Kleisli category $\mathbf{PCoh}_{!_a}$.

$\mathbf{PCoh}_{!_a}(\mathcal{A},\mathcal{B}) = \mathbf{PCoh}(!_a\mathcal{A},\mathcal{B})$: *analytical functions* from $\mathrm{P}(\mathcal{A})$ to $\mathrm{P}(\mathcal{B})$.

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$M = $ let rec $f\,x\,y =$
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$u \in (\mathbb{R}^+)^{|!_a\mathrm{Bool} \multimap 1|}$:

$$u_{\mu,\star} = \begin{cases} 4 \text{ if } \mu = [\mathtt{t}, \mathtt{f}] \\ 0 \text{ otherwise.} \end{cases}$$

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Fact:

$v \in (\mathbb{R}^+)^{|!_aX|\times|Y|}$:

$$v \in \mathbf{PCoh}_{!_a}(X,Y) \Leftrightarrow \forall x \in \mathrm{P}(X),\ vx^! \in \mathrm{P}(Y).$$

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$$[\![M]\!](x,y) = [\![M]\!] \cdot x^! \cdot y^! = \sum_{n\in\mathbb{N}} \sum_{\substack{(b_1,\ldots,b_n),(a_1,\ldots,a_n)\\ |b_i\neq a_i \wedge b_n = a_n}} \prod_{i=1}^{n} x_{a_i} y_{b_i}$$

Example (Not Definable Element in $\mathrm{P}(!_a\mathrm{Bool} \multimap 1)$.)

$u \in (\mathbb{R}^+)^{|!_a\mathrm{Bool}\multimap 1|}$:

$$u_{\mu,\star} = \begin{cases} 4 \text{ if } \mu = [\mathtt{t},\mathtt{f}] \\ 0 \text{ otherwise.} \end{cases}$$

Fact:

$v \in (\mathbb{R}^+)^{|!_a X| \times |Y|}$:

$v \in \mathbf{PCoh}_{!_a}(X,Y) \Leftrightarrow \forall x \in \mathrm{P}(X),\ vx^! \in \mathrm{P}(Y)$.

For $x \in \mathrm{P}(\mathrm{Bool})$:

$$(ux^!)_\star = 4x_{\mathtt{t}}x_{\mathtt{f}} \leq 1 \text{ since } x_{\mathtt{t}} + x_{\mathtt{f}} \leq 1$$

The Kleisli category $\mathbf{PCoh}_{!_a}$.

$\mathbf{PCoh}_{!_a}(\mathcal{A},\mathcal{B}) = \mathbf{PCoh}(!_a\mathcal{A},\mathcal{B})$: *analytical functions* from $\mathrm{P}(\mathcal{A})$ to $\mathrm{P}(\mathcal{B})$.

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The Kleisli category $\mathbf{PCoh}_{!_a}$.

$\mathbf{PCoh}_{!_a}($ … $)$ … *analytical functions* from … to …

Example

$M =$ … y_{b_i}

Example

$u \in \mathrm{P}(!$ …

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The Kleisli category $\mathbf{PCoh}_{!_a}$.

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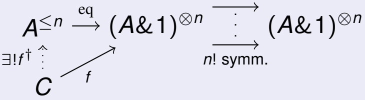
$\Rightarrow$ We try to construct it.

Equalizer of n-symmetries

$(A^{\leq n}, eq)$ such that:
$\forall C, f : C \to (A\&1)^{\otimes n}$ invariant by symmetries



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$$\begin{array}{ccccc} A^{\leq n} & \xrightarrow{\text{eq}} & (A\&1)^{\otimes n} & \underset{n!\ \text{symm.}}{\overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}}} & (A\&1)^{\otimes n} \\ \exists! f^{\dagger} \uparrow & \nearrow_{f} & & & \\ C & & & & \end{array}$$

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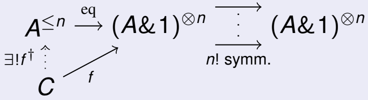

In **PCoh**: Equalizer $\mathcal{A}^{\leq n}$ of a probabilistic coherence space $\mathcal{A}$.

$$|\mathcal{A}^{\leq n}| = \mathcal{M}_{\leq n}(|\mathcal{A}|)$$
$$\mathrm{P}\left(\mathcal{A}^{\leq n}\right) = \{\langle u_1, \ldots, u_n \rangle \mid \forall i, u_i \in \mathrm{P}(\mathcal{A})\}^{\perp\perp}$$
$$\langle u_1, \ldots, u_n \rangle_{[a_1, \ldots, a_k]} = \frac{1}{\# S_n} \sum_{\sigma \in S_n} \prod_{i=1}^{k} (u_{\sigma(i)})_{a_i}$$

$$eq_{\mu,(a_1,\ldots,a_n)} = \begin{cases} 1 \text{ if } \mu = [a_i \mid a_i \neq \star] \\ 0 \text{ otherwise.} \end{cases}$$

Melliés, Tabareau, Tasson's recipe (1)

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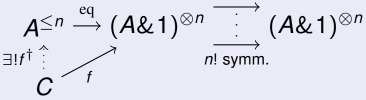

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Example ($\mathrm{Bool}^{\leq 2}$)

- $|\mathrm{Bool}^{\leq 2}| = \mathcal{M}_{\leq 2}(\{\mathtt{t}, \mathtt{f}\}) = \{[], [\mathtt{t}], [\mathtt{f}], [\mathtt{t}, \mathtt{t}], [\mathtt{t}, \mathtt{f}], [\mathtt{f}, \mathtt{f}]\}$
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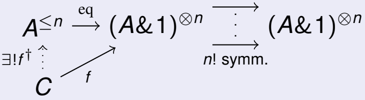

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Remark:

$\langle e_{\mathtt{t}}, e_{\mathtt{f}}\rangle \notin \mathrm{P}\left(!_a \mathrm{Bool}\right)$
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Melliès, Tabareau, Tasson's receipt (2)

Theorem (Melliès,Tabareau,Tasson)

In a cartesian symmetric monoidal category, where:

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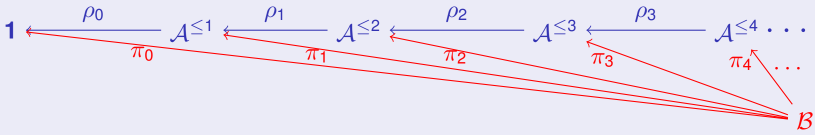

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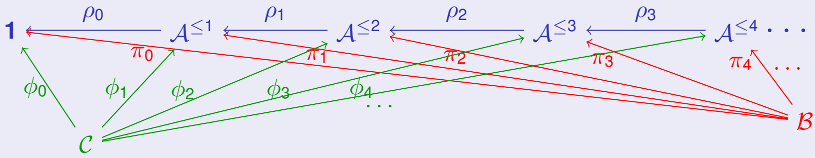

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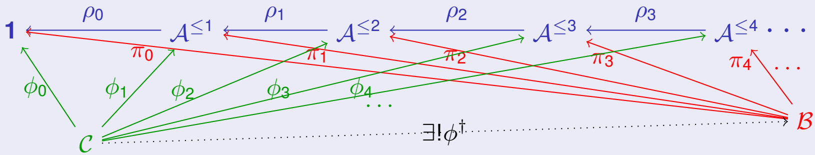

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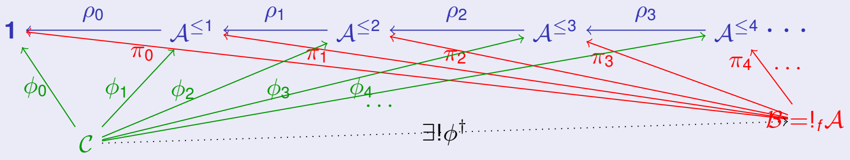

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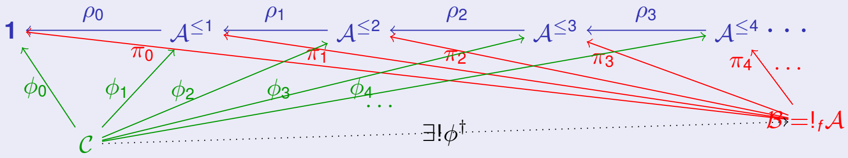

Then $\mathcal{B}$ is the free commutative comonoid $!_f\mathcal{A}$ of $\mathcal{A}$.

In **PCoh**.

- The $\mathcal{A}^{\leq n}$ exist and they commute with $\otimes$.
- Does the diagram admit a limit ?

Applying the recipy to PCoh

Constructing the limit $\mathcal{B}$ in **PCoh**

$$\mathbf{1} \xleftarrow{\rho_0} \mathcal{A}^{\leq 1} \xleftarrow{\rho_1} \mathcal{A}^{\leq 2} \xleftarrow{\rho_2} \mathcal{A}^{\leq 3} \xleftarrow{\rho_3} \mathcal{A}^{\leq 4} \cdots$$

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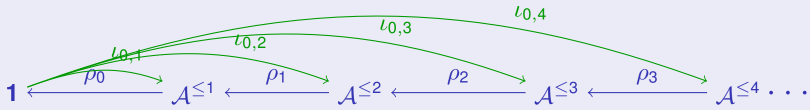

Proof summary:

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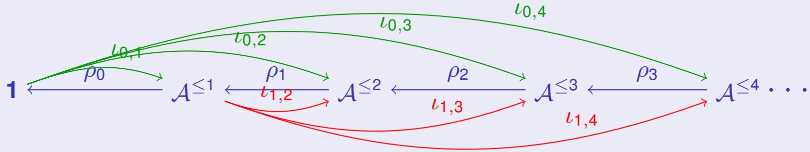

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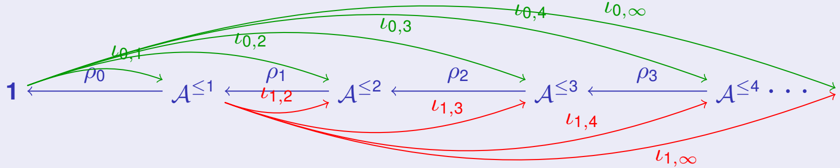

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Applying the recipy to PCoh

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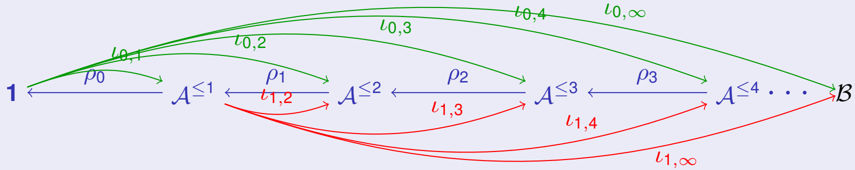

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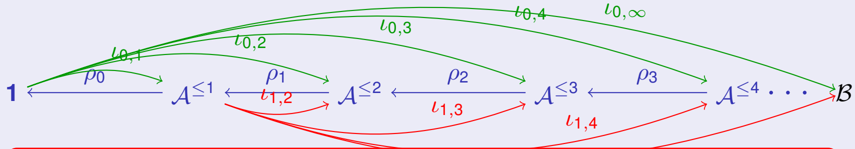
Definition

$\mathcal{B} = \{\iota_{n,\infty}(\langle u_1, \ldots, u_n\rangle) \mid n \in \mathbb{N}, \forall i, u_i \in \mathcal{A}\}^{\perp\perp}$.

Applying the recipy to PCoh

Constructing the limit $\mathcal{B}$ in **PCoh**



Definition of the $\iota_{n,N}$

Can we take :

$$(\iota_{n,N})_{\mu,\nu} \overset{?}{=} (\rho_{n,N}^{-1})_{\mu,\nu} = \begin{cases} 1 \text{ if } \mu = \nu \text{ and } \#\mu \leq n \\ 0 \text{ otherwise} \end{cases}$$

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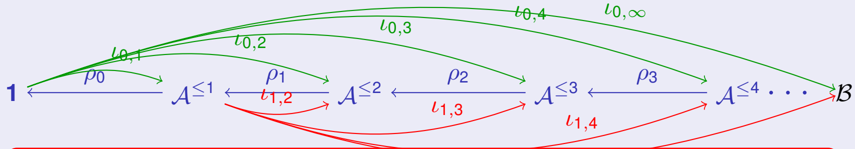

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$\notin$ **PCoh**$(\mathcal{A}^{\leq n}, \mathcal{A}^{\leq N})$

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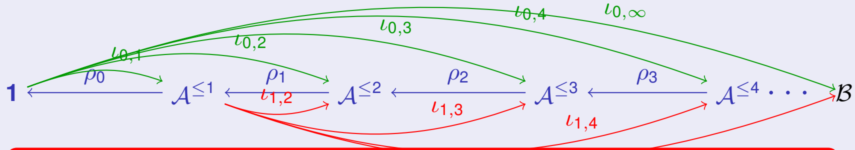
Constructing the limit $\mathcal{B}$ in **PCoh**



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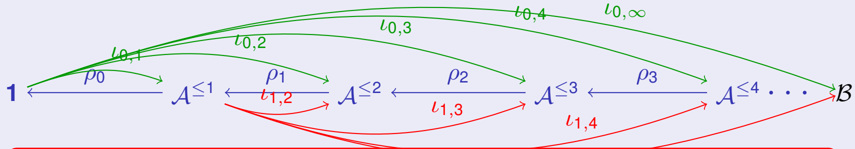
$$\rho_{2,5}^{-1}(\langle e_{\mathtt{t}}, e_{\mathtt{f}} \rangle) \notin \mathrm{P}\left(\mathrm{Bool}^{\leq 5}\right),$$

$$\vdots$$

⊥⊥ .

Constructing the limit $\mathcal{B}$ in **PCoh**



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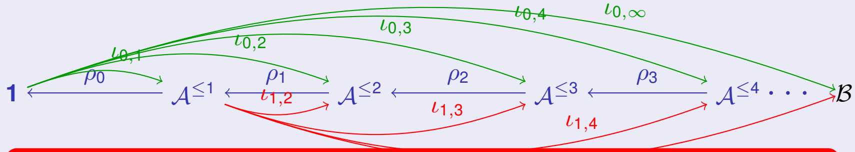
$$\rho^{-1}_{2,5}(\langle e_{\mathtt{t}}, e_{\mathtt{f}} \rangle) \notin \mathrm{P}\left(\mathrm{Bool}^{\leq 5}\right), \quad \text{but } \frac{2}{5}\rho^{-1}_{2,5}(\langle e_{\mathtt{t}}, e_{\mathtt{f}} \rangle) \in \mathrm{P}\left(\mathrm{Bool}^{\leq 5}\right)$$

$$\vdots$$

Constructing the limit $\mathcal{B}$ in **PCoh**



Pro

⊥⊥.

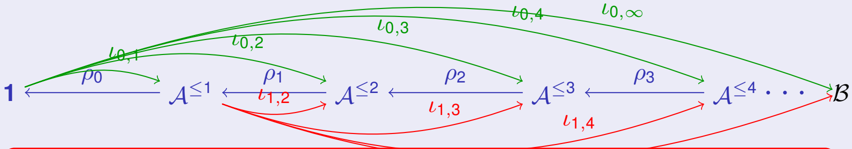
Definition of the $\iota_{n,N}$

$$(\rho_{n,N}^{-1})_{\mu,\nu} = \begin{cases} 1 \text{ if } \mu = \nu \text{ and } \#\mu \leq n \\ 0 \text{ otherwise} \end{cases} \quad \notin \mathbf{PCoh}(\mathcal{A}^{\leq n}, \mathcal{A}^{\leq N})$$

$$(\iota_{n,N})_{\mu,\nu} = \begin{cases} \frac{(N-k)!\lfloor N/n \rfloor^k n!}{N!(n-k)!} & \text{if } \mu = \nu, \#\mu = k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Applying the recipy to PCoh

Constructing the limit $\mathcal{B}$ in **PCoh**



Pro

$\perp\perp$.

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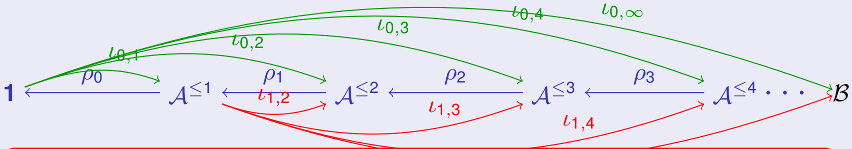

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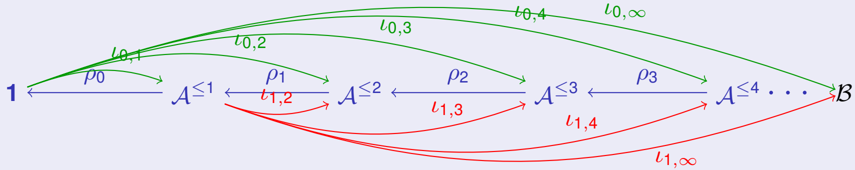
$(\iota_{n,N})_{\mu,\nu}$ has a limit for $N \to \infty$.

Applying the recipy to PCoh

Constructing the limit $\mathcal{B}$ in **PCoh**



Proof summary:

- $\iota_{n,N} \in \mathbf{PCoh}(\mathcal{A}^{\leq n}, \mathcal{A}^{\leq N})$, $\forall N > n$.
- $\iota_{n,\infty} \in \mathbb{R}_+^{\mathcal{M}_{\leq n}(|\mathcal{A}|)\times\mathcal{M}_f(|\mathcal{A}|)}$ as: $(\iota_{n,\infty})_{\mu,\nu} = \lim_{N\to\infty} \iota_{n,N}$

Definition

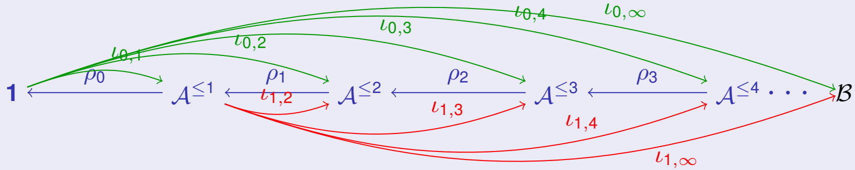
$$\mathcal{B} = \{\iota_{n,\infty}(\langle u_1, \ldots, u_n\rangle) \mid n \in \mathbb{N}, \forall i, u_i \in \mathcal{A}\}^{\perp\perp}.$$

Proposition

- $\mathcal{B}$ is the limit of the diagram above;
- it commutes with the tensor product.

Applying the recipy to PCoh

Constructing the limit $\mathcal{B}$ in **PCoh**



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$\Rightarrow$ We can apply Melliès-Tabareau-Tasson Theorem

The end of the story

Theorem: The "free" exponential modality

$$|!_f\mathcal{A}| = \mathcal{M}_{\mathrm{f}}(|\mathcal{A}|), \qquad \mathrm{P}(!_f\mathcal{A}) = \{\iota_{n,\infty}(\langle u_1, \ldots, u_n\rangle) \mid u_i \in \mathrm{P}(\mathcal{A}), n \in \mathbb{N}\}^{\perp\perp}$$

with $\iota_{n,\infty}(\langle u_1, \ldots, u_n\rangle)_{[a_1,\ldots,a_k]} = \frac{1}{n^k}\sum_{f:k\hookrightarrow n}\prod_{i=1}^{k}(u_{f(i)})_{a_i}$.

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The “analytic” exponential modality — Danos&Ehrhard2011

$$|!_a\mathcal{A}| = \mathcal{M}_{\mathrm{f}}(|\mathcal{A}|), \quad \mathrm{P}(!_a\mathcal{A}) = \left\{u^! \mid u \in \mathrm{P}(\mathcal{A})\right\}^{\perp\perp}, \quad \text{with } u^!_{[a_1,\dots,a_k]} = \prod_{i=1}^{k} u_{a_i}.$$

The end of the story

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The “analytic” exponential modality Danos&Ehrhard2011

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Theorem

The “free” and “analytic” exponential modalities are the same, i.e. $\mathrm{P}(!_f\mathcal{A}) = \mathrm{P}(!_a\mathcal{A})$.

Proof.

- $\mathrm{P}(!_f\mathcal{A}) \subseteq \mathrm{P}(!_a\mathcal{A})$, because $\iota_{n,\infty}(\langle u_1, \ldots, u_n\rangle) \leq (\sum_{i=1}^{n}\frac{1}{n}u_i)^!$
- $\mathrm{P}(!_a\mathcal{A}) \subseteq \mathrm{P}(!_f\mathcal{A})$, because $u^! = \lim_n \iota_{n,\infty}(\langle \underbrace{u, \ldots, u}_{n}\rangle)$

□

Conclusion

Contribution:

- Giving a categorical justification of $!_a$.
- Allows a decomposition of the universal property of $!_a$.
- Use of Mellìes,Tabareau,Tasson recipy as a tool to show that an exponential is free.

Conclusion

Contribution:

- Giving a categorical justification of $!_a$.
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Further work: The definability problem

Definability: every element is upper bounded by a definable element.

Not true in **PCoh**: $u \in \mathrm{P}\left(!_a \mathrm{Bool} \multimap 1\right)$ defined by:

$$u_\mu = \begin{cases} 4 \text{ if } \mu = [\mathtt{t}, \mathtt{f}] \\ 0 \text{ otherwise.} \end{cases}$$

$\forall M$ term of probabilistic *PCF* of type $\mathrm{Bool} \rightarrow 1$, $[\![M]\!]_{[\mathtt{t},\mathtt{f}]} \leq 2$.