De Finetti Theorem and Models of Linear Logic

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Two Categorical Approaches to Probability Theory

Synthetic probability theory

- Markov categories: categorical axiomatic for probability theory;
- Objective: state and prove abstractly the counterparts in Markov categories of textbook results from probability theory.

[Cho-Jacobs'19], [Fritz'20]...

Semantics of (higher-order) probabilistic programming languages

- Negative results:
  - **Meas** (measurable spaces, measurable functions) isn't cartesian closed;
  - **Stoch** (measurable spaces, stochastic kernels) isn't monoidal closed.
- Objective: build categories that behave *almost like* **Meas** or **Stoch**, but also have an internal Hom.

Linear Logic ([Ehrhard-Pagani-Tasson'18],[Ehrhard-Geoffroy'23],[Paquet'21]), topological domain-theory, Quasi-Borel spaces[Staton et al'17],...

Two (categorical) results on exchangeability

De Finetti theorem

- A foundational result from early probability theory:

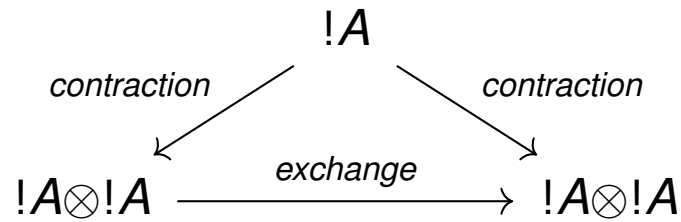
$$\mathrm{Meas}^{=1}([0,1]) \sim \left\{ \begin{array}{l} \mathbf{X} = X_1, \ldots, X_n, \ldots \\ \text{each } X_i : \{0,1\}\text{-valued random variable} \\ \mathbf{X} \text{ exchangeable} \end{array} \right\}$$

Kolmogorov world — discrete world

- Expressed as a categorical limit in [Jacobs-Staton'20], [Perrone et al'21]

Exchangeability in models of linear logic

- $!A$ is a commutative comonoid (due to Seely iso: $!A \otimes !B \sim !(A \,\&\, B)$):



- Lafont model: $!A$ is the free commutative comonoid
- [Melliès-Tabareau-Tasson'18]: a categorical recipe to build the free commutative comonoid in a symmetric monoidal category.

Categorical De Finetti Theorem in **Stoch**

Stoch: the category of Stochastic kernels

The Giry monad on **Meas**

$\mathscr{G}X := \mathrm{Meas}^{=1}(X)$, equiped with the structure of a measurable space (i.e. a σ-algebra).

- $x \in X \mapsto \delta_x \in \mathscr{G}X$
- $\mathscr{D} \in \mathscr{G}\mathscr{G}X \mapsto (A \in \Sigma_X \mapsto \int_{\mu \in \mathscr{G}X} \mu(A) \cdot d\mathscr{D} \in \mathbb{R}_{\geq 0}) \in \mathscr{G}X$.

Definition

Stoch is the Kleisli category of the Giry monad:

$$\mathbf{Stoch}(X, Y) = \{X \to \mathscr{G}Y \in \mathbf{Meas}\}$$

Proposition

*The cartesian product in **Meas** becomes a monoidal product in **Stoch**. ($\mathscr{G}$ is a strong monad).*

Standard Borel spaces.

Separable, completely metrizable topological spaces equipped with their Borel σ-algebra. In this talk: **Meas** := **SBS**; **Stoch** := **Stoch**(**SBS**).

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Exchangeable Sequences of Random Variables:

(Ω, μ) a probabilistic space.

- Random variable over A: a map $X : \Omega \to A$ measurable
- $\text{law}(X) \in \Delta(A) := \{\text{probability distributions over } A\}$

Definition

An infinite list $\vec{X} = X_1, X_1, ...X_n, ...$ of A-valued random variables is exchangeable, when $\forall n \in \mathbb{N}$, $\forall$ permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$:

$$\text{law}(X_1, \ldots, X_n) = \text{law}(X_{\sigma(1)}, \ldots, X_{\sigma(n)}) \qquad \in \Delta(A^n)$$

Example

An infinite sequence of independent, identically distributed (i.i.d) A-valued random variables is always exchangeable:

$$\begin{aligned}\text{law}(X_1, \ldots, X_n) =& \text{law}(X_1) \times \ldots \times \text{law}(X_n) \text{ by independence} \\ =& \mathscr{D} \otimes \ldots \otimes \mathscr{D} \text{ where } \mathscr{D} = \text{law}(X_1) = \text{law}(X_2) = \ldots\end{aligned}$$

Mixtures of i.i.d sequences

Definition (Discrete mixture)

$\mathbf{X}^1, \ldots, \mathbf{X}^m$ a finite family of sequences of A-valued random variables.
A *mixture* of the $\mathbf{X}^1, \ldots, \mathbf{X}^m$ along a $\mathscr{D} \in \Delta(\{1, \ldots, m\})$ is any sequence of random variables $\mathbf{Y}$ such that:

$$\forall n \in \mathbb{N},\ \text{law}(Y_1, \ldots, Y_n) = \sum_{j=1}^{m} \mathscr{D}(j) \cdot \text{law}(X_1^j, \ldots, X_n^j).$$

$\mathbf{Y}$ should be thought of as:

- first, an index j is chosen in $\{1, \ldots, m\}$ along $\mathscr{D}$;
- then $\mathbf{Y}$ behaves as $\mathbf{X^j}$ for every step $n \in \mathbb{N}$.

Lemma

- *A mixture of exchangeable sequences is always exchangeable.*
- *A mixture of i.i.d sequences is an* $\mathbf{Y} = Y_1, \ldots, Y_n \ldots$ *such that* $\exists \mathscr{D} \in \Delta(\{1, \ldots, m\}), \exists (\mathscr{E}^j)_{1 \leq j \leq m}$ *with:*

$$\forall n \in \mathbb{N},\ \textit{law}(Y_1, \ldots, Y_n) = \sum_{j=1}^{m} \mathscr{D}(j) \cdot \mathscr{E}^j \otimes \ldots \otimes \mathscr{E}^j.$$

De Finetti Theorem

Theorem (De Finetti)

Under some conditions on A, any infinite exchangeable A-valued sequences $X_1, \ldots, X_n, \ldots$ can be seen as continuous mixtures of i.i.d sequences:

$$\exists D \in \mathscr{G}\mathscr{G}A \text{ s.t. } \forall n, law(X_1, \ldots, X_n) = \int_{\mu \in \mathscr{G}A} (\mu \otimes \ldots \otimes \mu) \cdot D(d\mu).$$

Theorem (Corollaire)

There is a bijection between the exchangeable measures on A^ω ($:=$ the laws of some infinite exchangeable sequence of A-rand.var.) and $\mathscr{G}\mathscr{G}A$.

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In particular, if X is a standard Borel space.

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Jacobs-Staton Categorical Formalisation (1)

Definition (The Draw-and-Delete Chain in **Stoch**)

$$\mathbf{1} \xleftarrow{D_0} \mathcal{M}_1(A) \xleftarrow{D_1} \mathcal{M}_2(A) \xleftarrow{D_2} \cdots$$

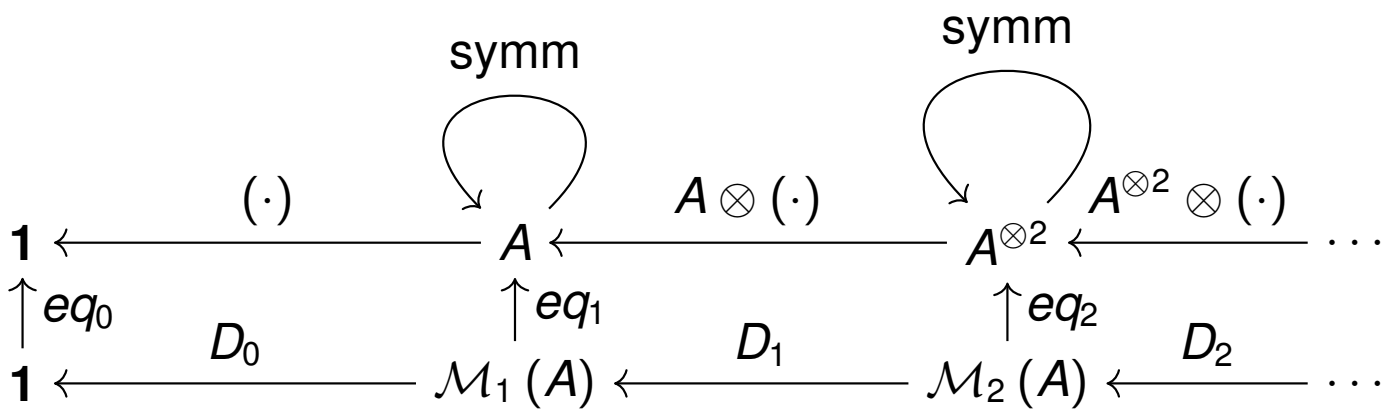
$\mathcal{M}_n(A)$: set of multiset of size n (over A)

Draw-and-delete morphism $\mathrm{DD}_n : \mathcal{M}_n(A) \to \mathscr{G}(\mathcal{M}_{n-1}(A))$.

- takes in input an urn with n elements;
- draw at random an element in the urn, and discard it.

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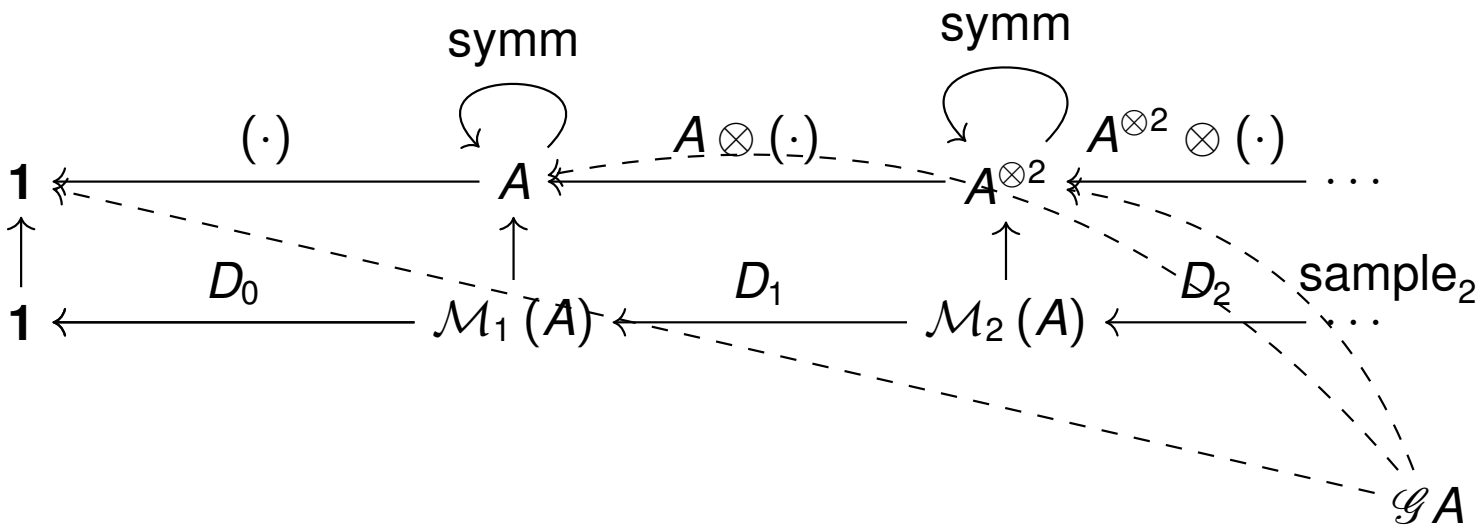
- takes in input an urn with n elements;
- draw at random an element in the urn, and discard it.

$(\cdot) : A \to \mathbf{1}$ $\quad$ $\mathbf{1} = \{\star\}$ is terminal in **Stoch** ($\mathscr{G}(\{\star\}) = \{\delta_\star\}$).

Jacobs-Staton Categorical Formalisation (2)

Theorem

*The measurable space $\mathscr{G}A$ is the categorical limit in **Stoch** of the draw-and-delete chain.*

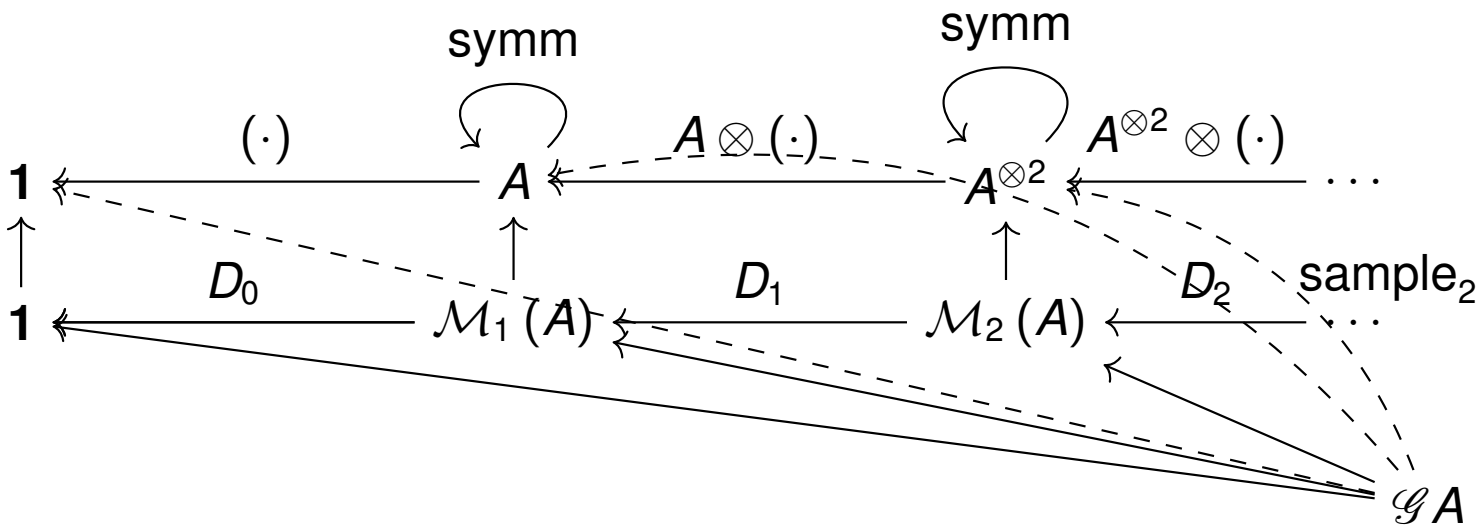


sample$_n$: $\mu \in \mathscr{G}A \to \mu \otimes \ldots \otimes \mu \in \mathscr{G}(A^n)$ in **Meas**.

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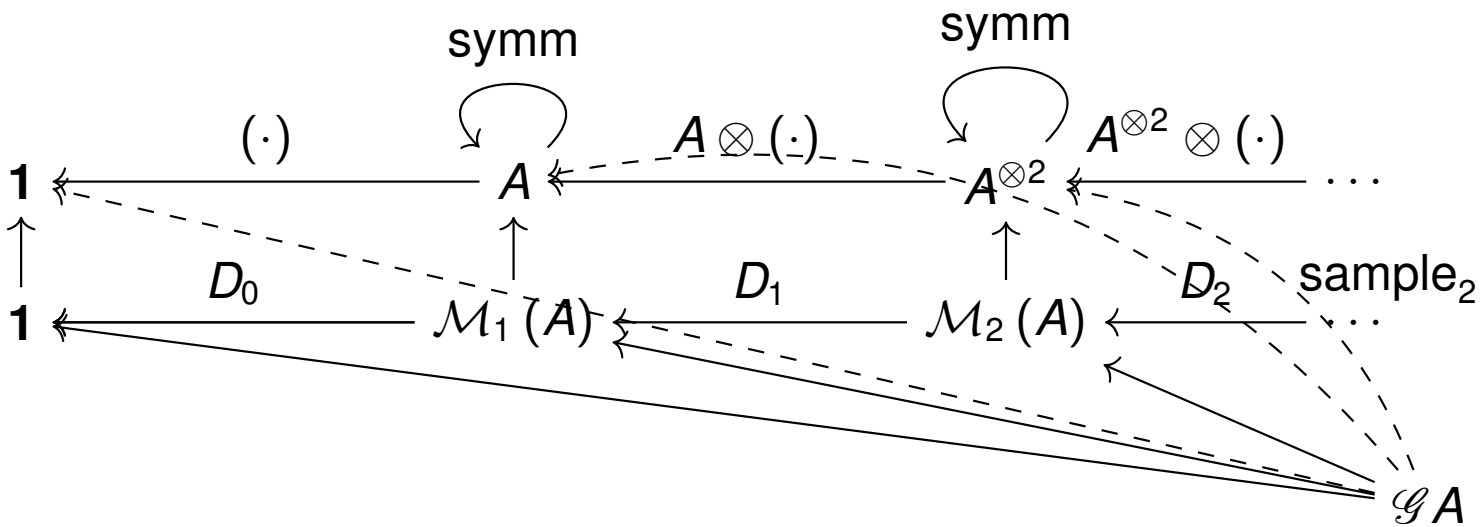


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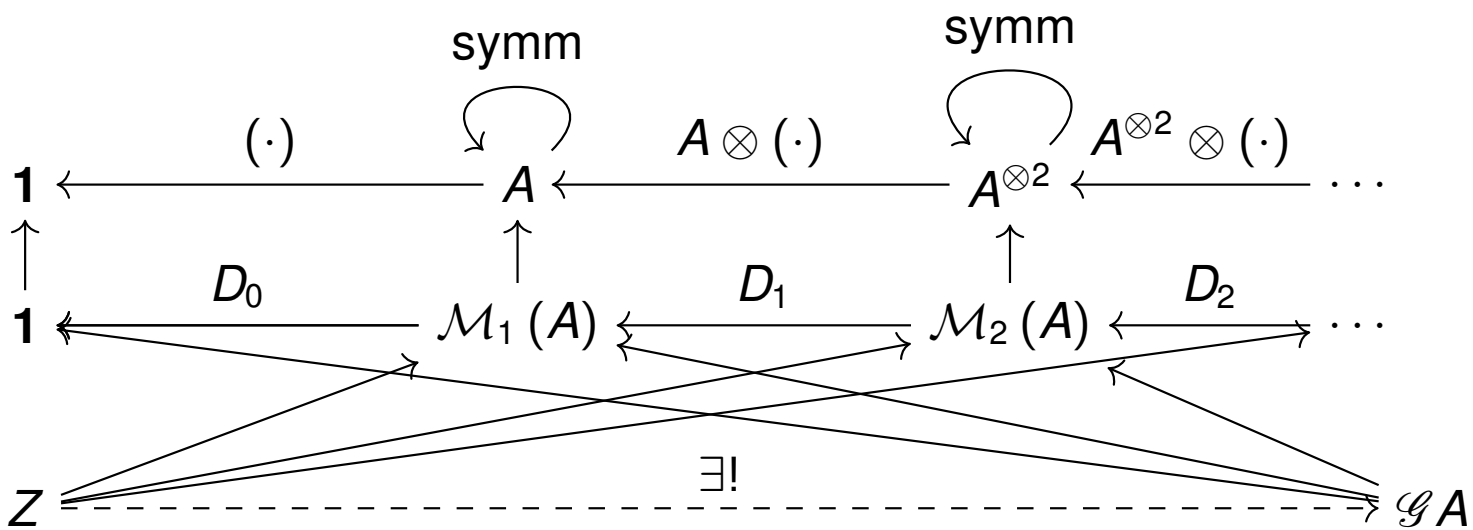
Remark:

- Any sequence $\mathbf{X} = X_1, \ldots, X_n, \ldots$ of random variable from (Ω, μ) to A gives a cone on the delete chain: $X_1, \ldots, X_n : \Omega \to A^{\otimes n}$ in **Meas**.
- If $\mathbf{X}$ is exchangeable, precomposing (in **Stoch**) this cone by $\mathbf{1} \xrightarrow{\mu} \Omega$ equalises all symmetries on $A^{\otimes n}$.

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Perrone et al Categorical De Finetti theorem

Theorem (Fritz et al'20)

The delete chain itself has a limit A^ω :

$$\mathbf{1} \xleftarrow{(\cdot)} A \xleftarrow{A \otimes (\cdot)} A^{\otimes 2} \xleftarrow{A^{\otimes 2} \otimes (\cdot)} \cdots$$

A^ω is the tensor product of infinitely many copies of A.

Proof.

Using Kolmogorov extension theorem, which holds for standard Borel spaces. □

Theorem (Perrone et al,)

*Under some conditions on A, $\mathscr{G}A$ is the equaliser in **Stoch** of the finitely supported symmetries on A^ω .*

Remark

- A sequence **X** of random variables from (Ω, μ) to A gives a morphism $\Omega \to A^\omega$.
- When **X** is exchangeable, this morphism precomposed by $\mathbf{1} \xrightarrow{\mu} \Omega$, equalises all finitely supported symmetries on A^ω.

The layered construction of the free commutative comonoid

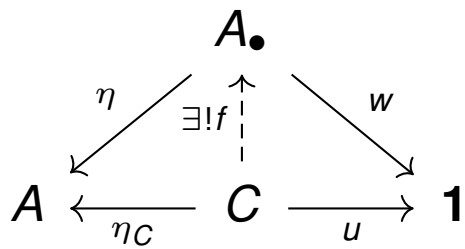
Méliès-Tabareau-Tasson Construction (1)

- A generic recipe to build the **free** commutative comonoid from any symetric monoidal category with enough structure.
- Layered construction: build $!A$ as the limit of approximations (spaces of ressources that can be used at most n times, in no particular order)

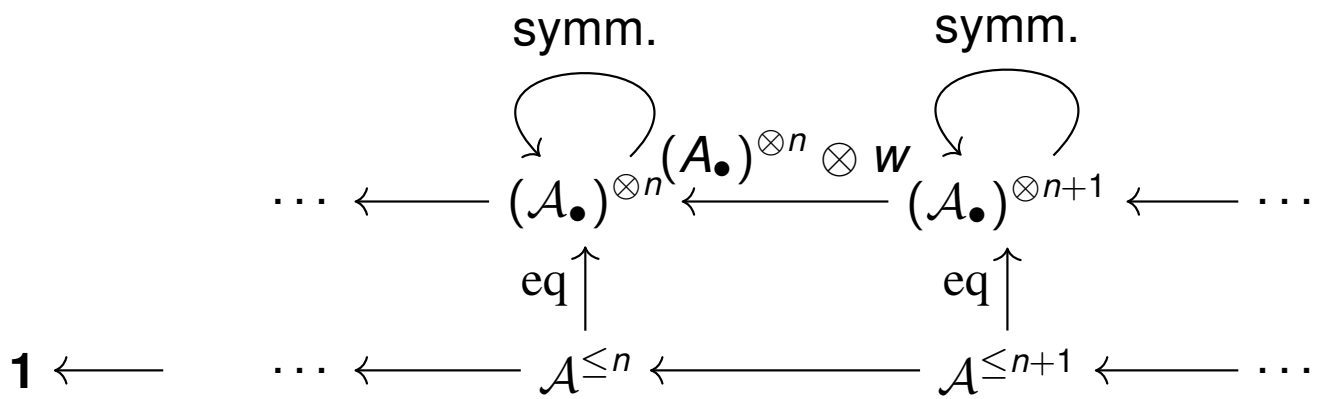
Free Copointed object generated by A

$$(A_\bullet, \eta : A_\bullet \to A, w : A_\bullet \to \mathbf{1})$$

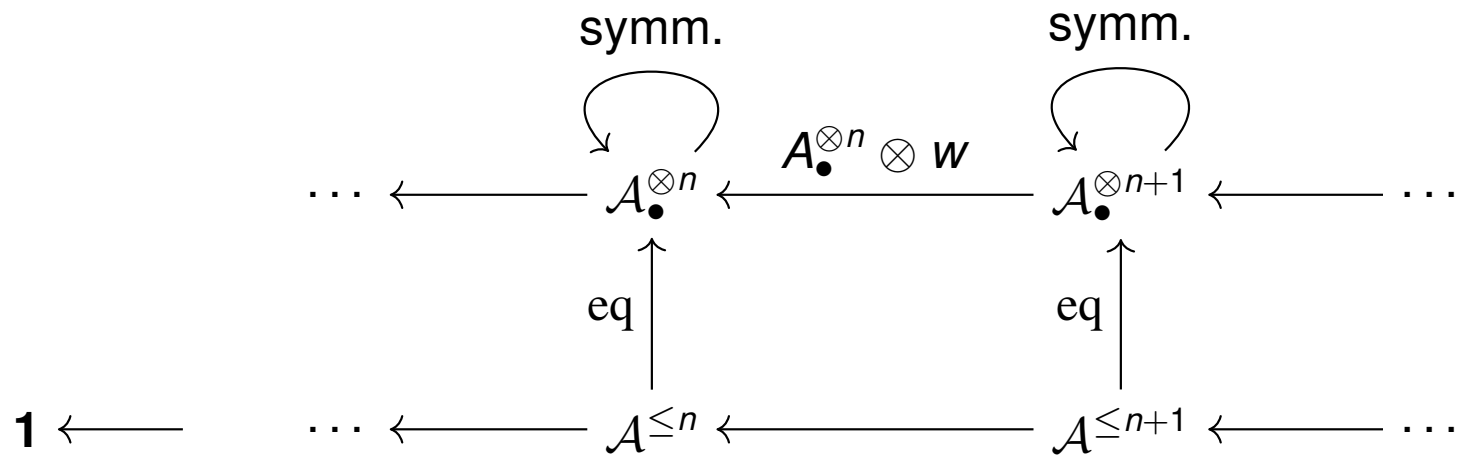
e.g $A_\bullet = A \,\&\, \mathbf{1}$ in presence of cartesian product
$A_\bullet = A$ if $\mathbf{1}$ is a terminal object.



The Approximations Chain



Méliès-Tabareau-Tasson Construction (2)



Theorem (Melliès-Tabareau-Tasson)

In any monoidal category $\mathbb{C}$ such that:

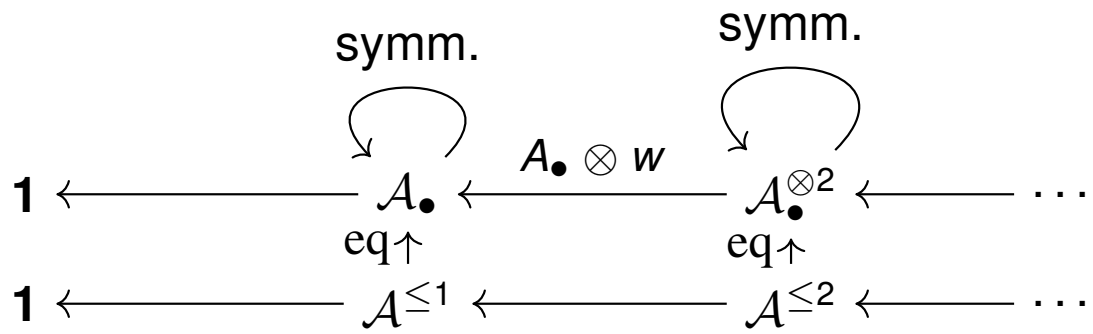
- *the free copointed object generated by A exists*
- *$\forall A$, $\forall n$, the equaliser of all symmetries over $A^{\otimes n}$ exist, and commutes with the tensor product,*
- *the approximant chain has a limit, and this limit commutes with the tensor product,*

Then the limit of the chain is the free commutative comonoid over A.
If moreover $\mathbb{C}$ is monoidal closed, it gives a Lafont model of Linear Logic.

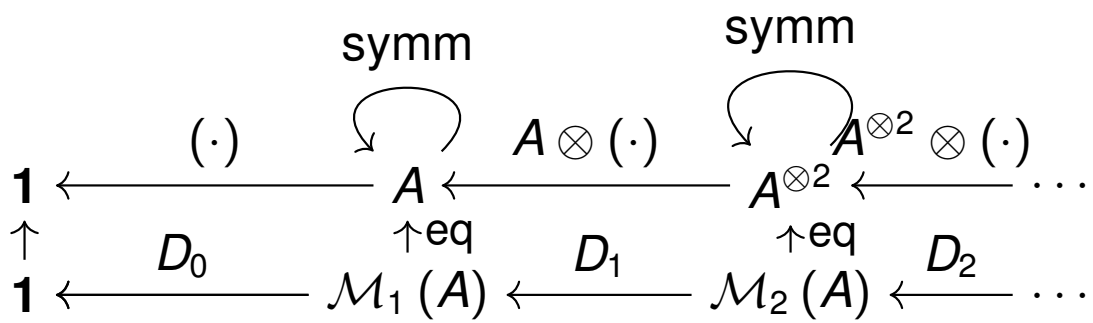
Connecting the two constructions in **Stoch**.

Comparing the two chains:

Approximations chain:



Draw-and-Delete chain:



Remark: Since the monoidal unit **1** is terminal in **Stoch**, the free copointed object over A in **Stoch** is simply A.

$\Rightarrow$ the approximation chain in **Stoch** generated by A is exactly the draw-and-delete chain of Jacobs-Staton's De Finetti theorem.

Commutation of [Perrone et al] De Finetti constructions under $\otimes$

Proposition (Fritz et al'20)

In ***Stoch****, the infinite tensor product commutes with* $\otimes$*, i.e.* $\forall A, B \in$ ***SBS****,* $A^\omega \otimes B$ *is the limit of the chain:*

$$B \xleftarrow{(\cdot)\otimes B} A \otimes B \xleftarrow{A\otimes(\cdot)\otimes B} A^{\otimes 2} \otimes B \xleftarrow{A^{\otimes 2}\otimes(\cdot)\otimes B} \cdots$$

Proposition (Parametrized De Finetti theorem)

In ***Stoch****, the equaliser of the finitely supported symmetries commutes with* $\otimes$*, i.e.* $\forall A, B, \mathscr{G}A \otimes B$ *is the equaliser of:*

$$A^\omega \otimes B \underset{\sigma'\otimes B}{\overset{\sigma\otimes B}{\rightrightarrows}} A^\omega \otimes B$$

Commutation of [Jacobs-Staton] De Finetti constructions under $\otimes$

Lemma

*In **Stoch**, the equaliser of symmetries on $A^{\otimes n}$ commutes with $\otimes$.*

Proposition

$\mathbb{C}$ a monoidal category where $\mathbf{1}$ is terminal, such that:

- *the infinite tensor product A^{ω} exists and commutes with $\otimes$;*
- *the equaliser of all symmetries on $A^{\otimes n}$ exist and commute with $\otimes$;*

The draw-and-delete chain has a limit iff the equaliser of the symmetries on A^{ω} exist, these limits coincide, and moreover if one commutes with $\otimes$ the other does too.

Corollary

*The limit of the draw and delete chain in **Stoch** commutes with $\otimes$.*

Theorem

*$\mathscr{G}A$ is the free commutative comonoid generated by A in **Stoch**.*

Proof: [Melliès-Tabareau-Tasson] theorem applied to **Stoch**.

De Finetti theorem and Exponential in probabilistic models of Linear Logic

Linear Logic and Probability Theory

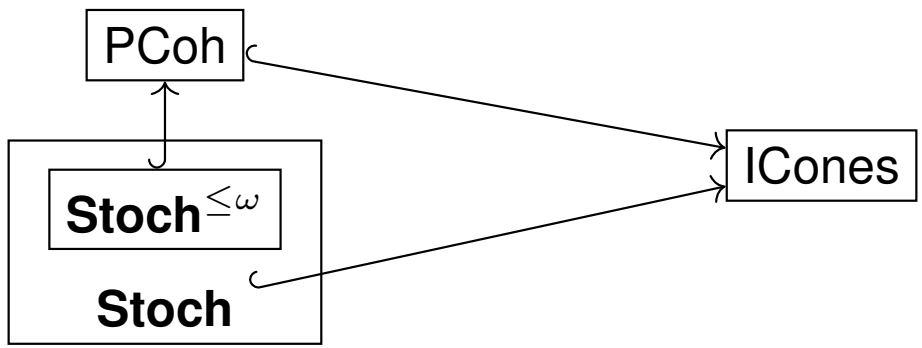
Remark: **Stoch** is not monoidal closed, thus cannot be a model of LL.

PCoh: Category of Probabilistic Coherence Space

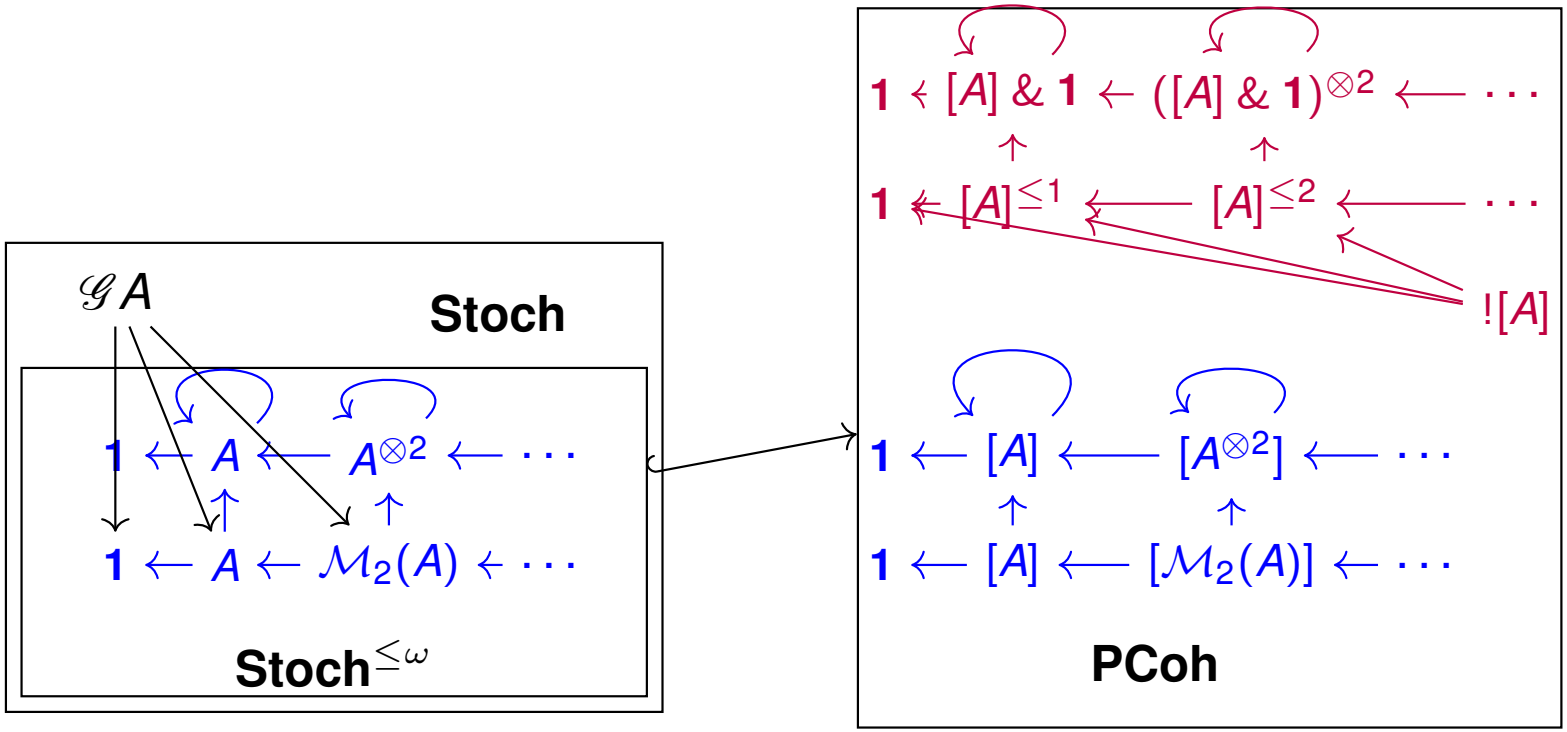
- model of Linear Logic;
- fully abstract model of discrete randomized PCF;
- ! is the free one, and can be built with the layered construction.

ICones: Category of Integrable Cones [Ehrhard-Geoffroy'23]

- model of Intuitionnistic Linear Logic
- model of continous randomized PCF.
- ! is the free one (but it is not known whether the layered construction hold).



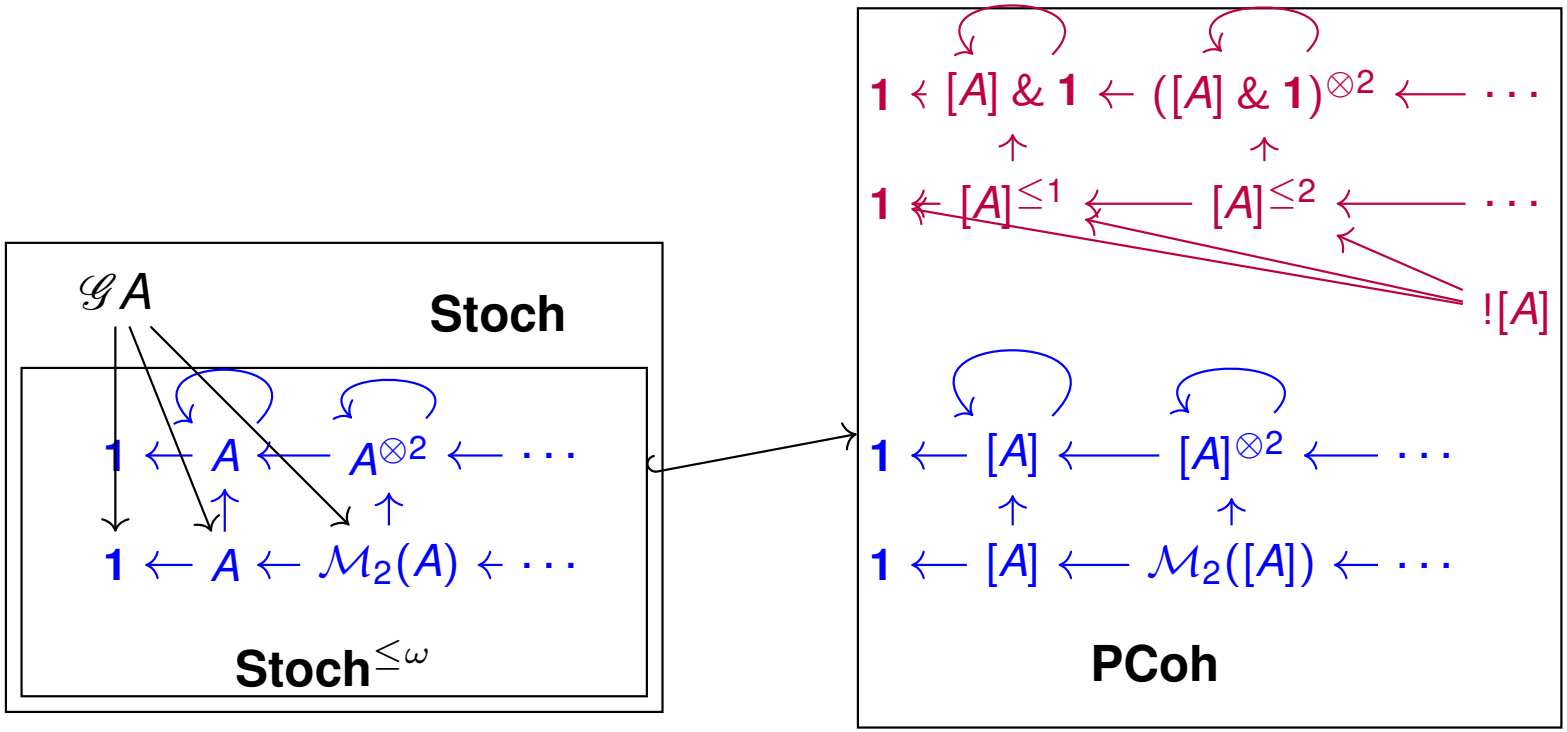
Draw-and-delete chains and Approximations chains in PCoh.



The functor $\mathbf{Stoch}^{\leq\omega} \hookrightarrow$ PCoh preserves:

- monoidal product;
- non-empty, connected limits.

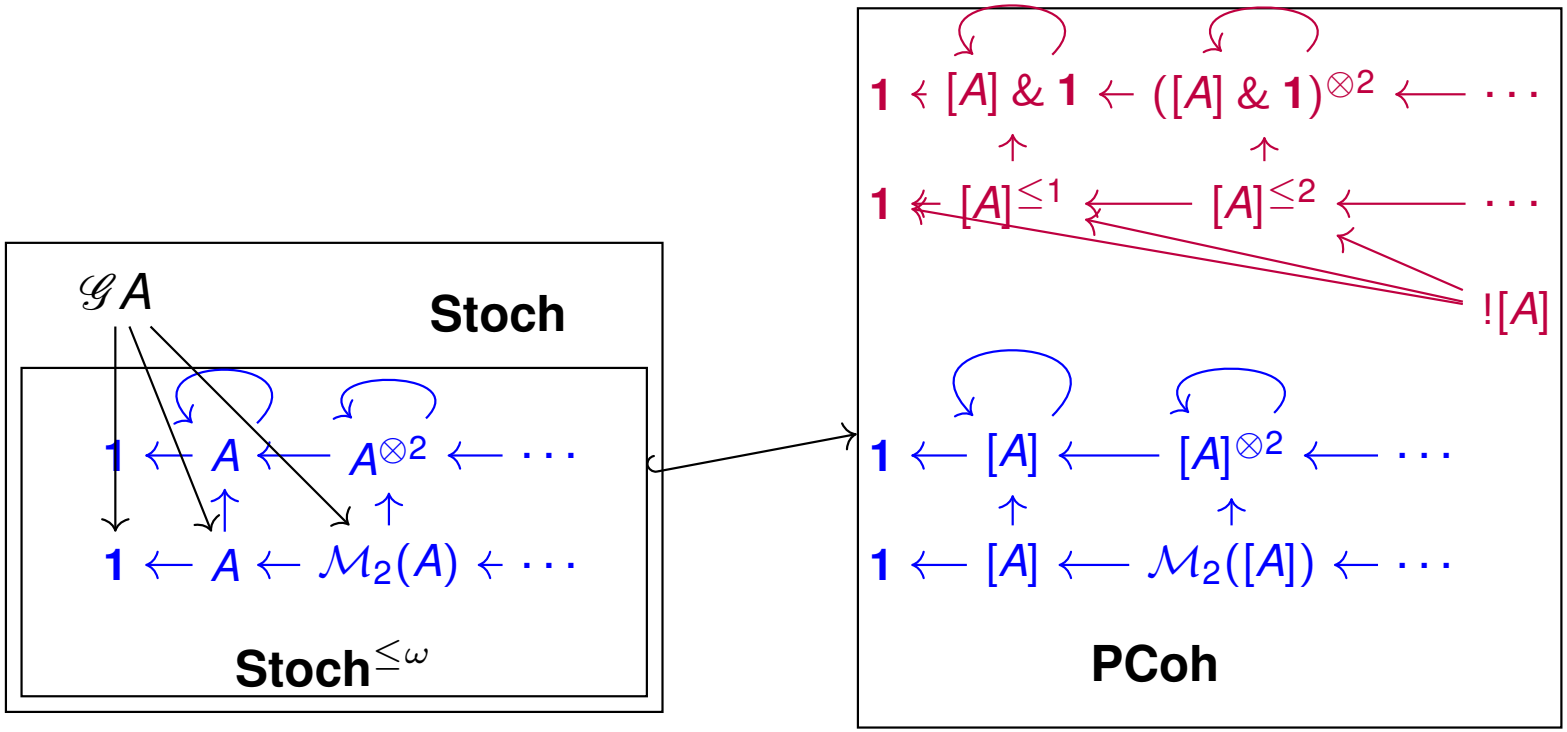
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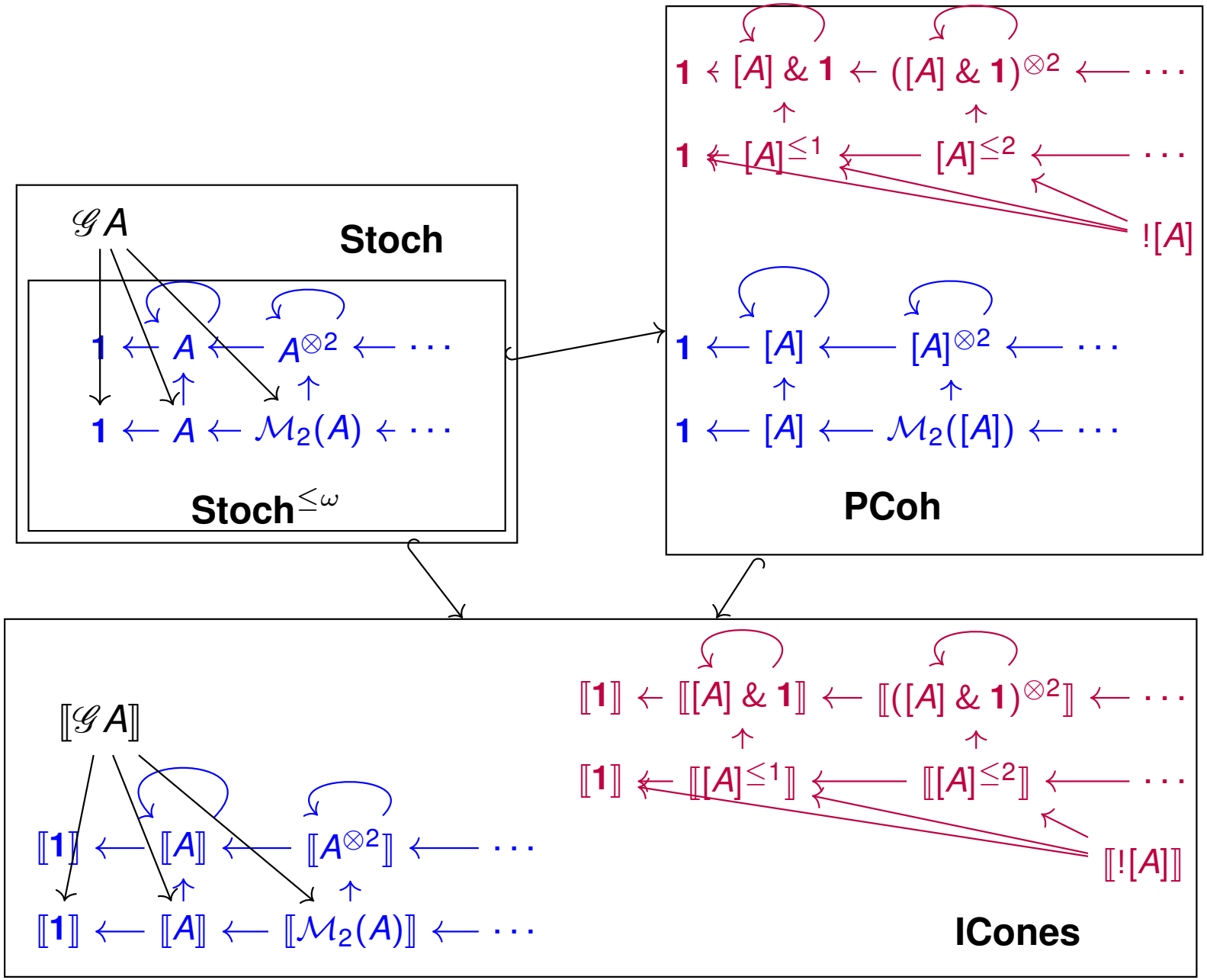
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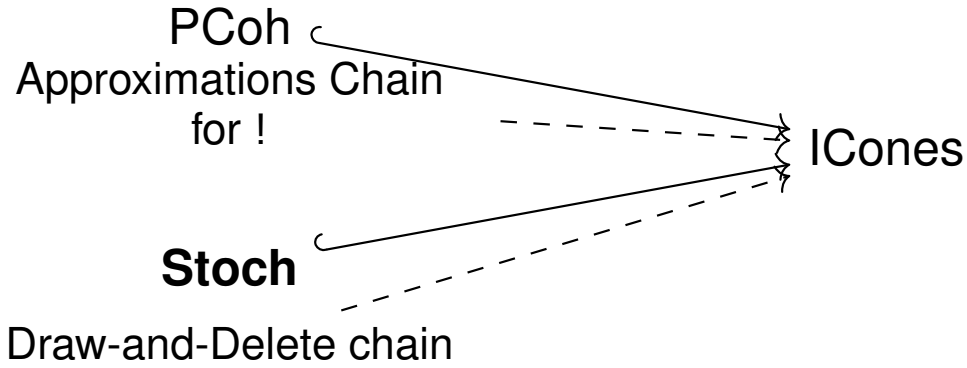
It does not preserve terminal object, and thus neither the free copointed object over A.

$\Rightarrow$ The draw-and-delete chain and the approximations chain do not coincide in PCoh.

Draw-and-delete chains and Approximations chains in **ICones**.



Transferring the chains to ICones



Theorem

The functor ***Stoch*** $\hookrightarrow$ *ICones preserves tensor product, and all (small) non-empty, connected limits.*

Theorem (Ehrhard-Geoffroy'23)

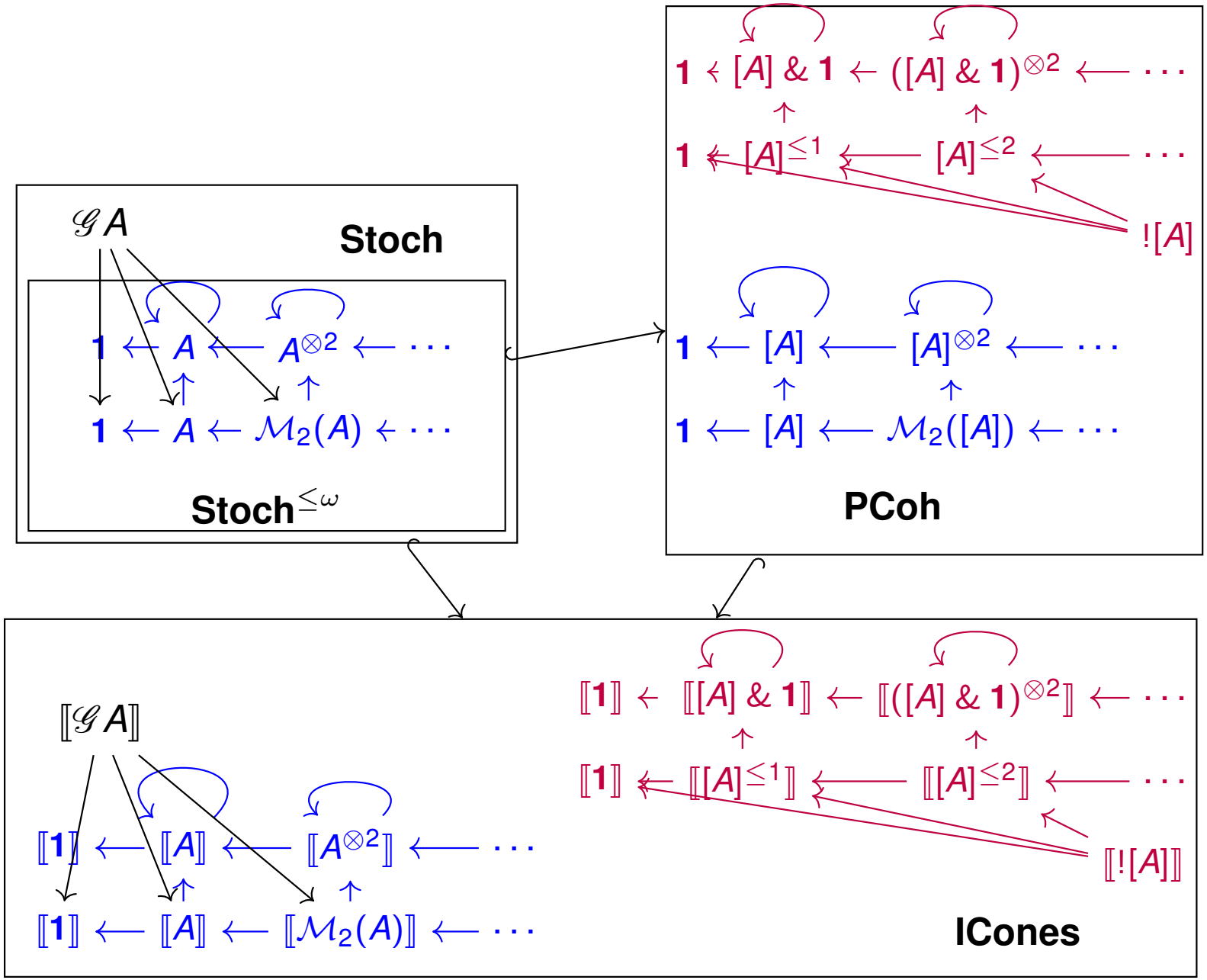
The functor PCoh $\hookrightarrow$ *ICones preserves tensor product and cartesian product.*

Open Question: Does PCoh $\hookrightarrow$ ICones preserves limits ?

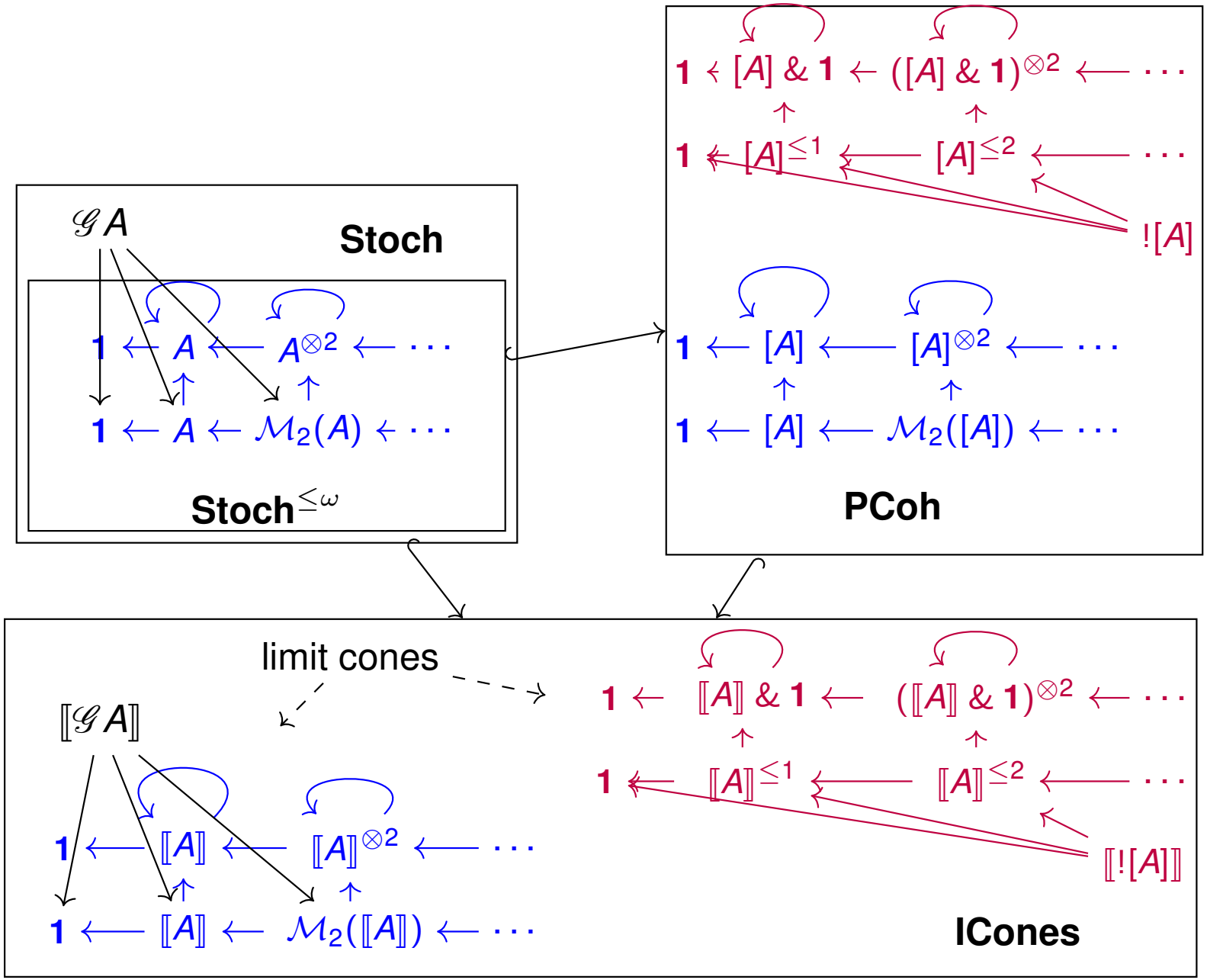
Proposition

$\forall X$ *countable discrete space in* $\mathbf{Stoch}^{\leq\omega}$, *the limit of the approximations chain for* $[X]$ *is preserved by PCoh* $\hookrightarrow$ *ICones.*

Draw-and-delete chains and Approximations chains in **ICones**.



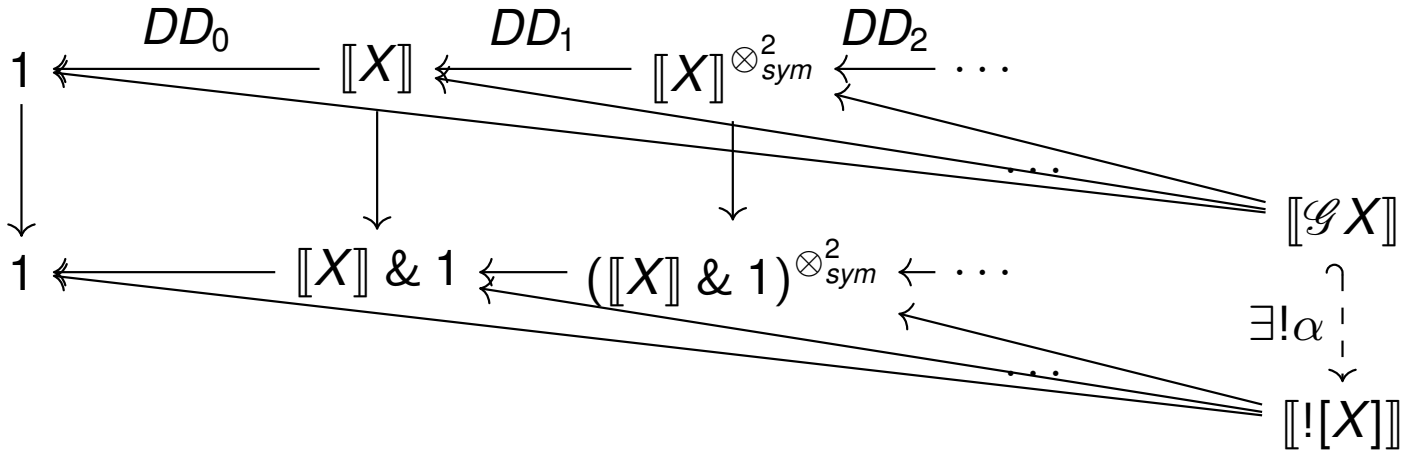
Draw-and-delete chains and Approximations chains in **ICones**.



Connecting the diagrams (in ICones):

Theorem

For X countable discrete space:



Proof.

- $[\![X]\!]$ & **1** is the free co-pointed object over $[\![X]\!]$;
- + limit caracterisation of $[\![![X]]\!]$.

□

Warning

There is **no chain morphism** in the **reverse** direction: from De Finetti chain to the approximations chain
(the approximations chain is **free** among all the chains with this shape).

Using α to caracterise the total elements in $!\llbracket X \rrbracket$.

Probabilistic coherence spaces (PCS)

pair $(|\mathcal{A}|, \mathrm{P}(\mathcal{A}))$ where:

- $|\mathcal{A}|$: *countable* web;
- $\mathrm{P}(\mathcal{A}) \subseteq \mathbb{R}_{\geq 0}^{|\mathcal{A}|}$: quantitative cliques/elements.

Orthogonality Relation

- For $u, v \in \mathbb{R}_{\geq 0}^{|\mathcal{A}|}$, $\langle u, v \rangle = \sum_{a \in |\mathcal{A}|} u_a \cdot v_a$;
- $\mathcal{A}^{\perp} = \{u \in \mathbb{R}_{\geq 0}^{|\mathcal{A}|} \mid \forall v \in \mathrm{P}(\mathcal{A}),\ \langle u, v \rangle \leq 1\}$.

Bi-Orthogonality Condition on PCSs

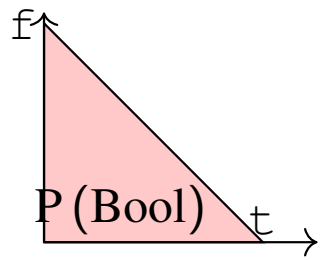
$\mathcal{A} = \mathcal{A}^{\perp\perp}$

The set of elements is downward, convex, Scott closed.

Example (Encoding a discrete data-type)

X countable discrete space:

- $|[X]| = X$;
- $\mathrm{P}(X) = \{u \in \mathbb{R}_{\geq 0}^{X} \mid \sum_{x \in X} u_x \leq 1\} \sim \mathscr{G}^{\leq 1} X$.

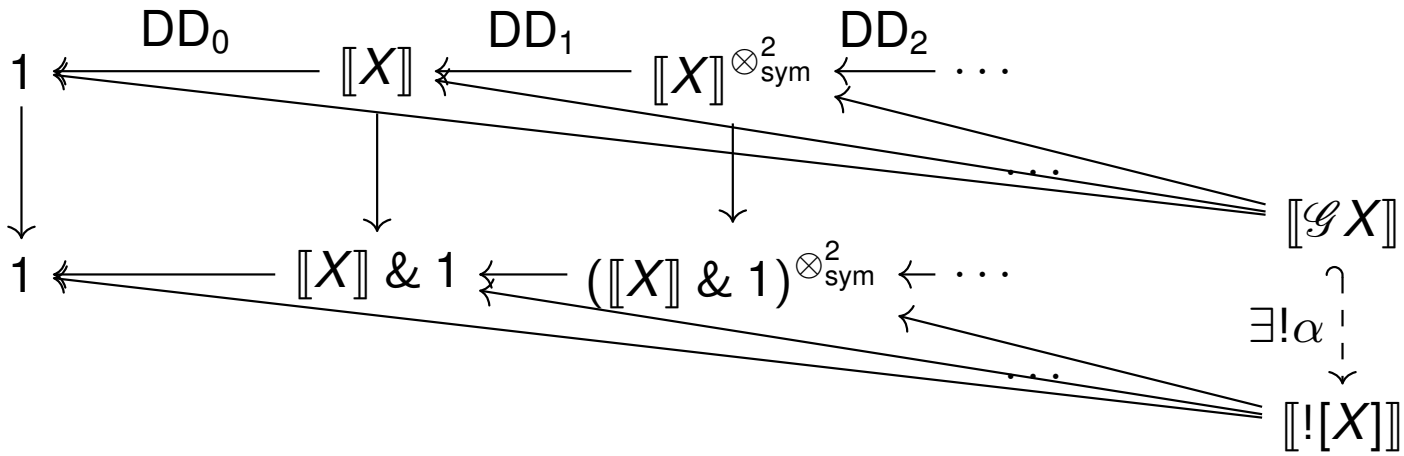


Exponential Comonad

- web: $|!\mathcal{A}| = \mathcal{M}_f(|\mathcal{A}|)$
- cliques: $\mathrm{P}(!\mathcal{A}) = \{x^{!} \mid x \in \mathrm{P}(\mathcal{A})\}^{\perp\perp}$
  where $x^{!}$ is the **promotion** of x: $x^{!}_{[a_1,\ldots,a_n]} := \prod_i x_{a_i}$.

Understanding α concretely:

For X countable discrete space:



$![X]$: space of denotations for processes that can produce an element of datatype X infinitely many times

What are the processes in the image of α like ?

As Set-function $\alpha : \mathscr{G}^{\leq 1}(\mathscr{G}X) \to \{\text{elements of } !X\}$.

1. the process is initialized by sampling in some continuous sub-probability distribution $\eta \in \mathscr{G}^{\leq 1}(\mathscr{G}X)$;
2. the process stores the result $s \in \mathscr{G}X$ of the sampling;
3. each time the process is called, it samples independently in s a x of type X.

$\Rightarrow$They are the continuous mixture of promotions $s^!$ with s an element in $[X]$.

Total Elements in $![X]$

Do we describe **all** elements in $![X]$ (in **pcoh**) by $\alpha : [\![\mathscr{G}X]\!] \to [\![![X]]\!]$? **No!**

- The image of α **does not capture** processes whose probability of answering decreases each time they are called.
- But we can write such processes in $![X]$.

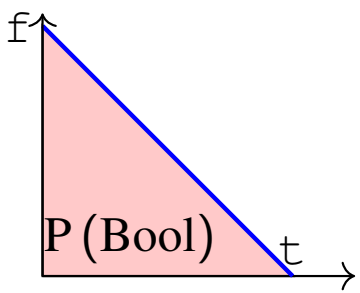
The model of proba.coherent spaces with totality [Ehrhard]

$(X, \mathrm{P}(X), \mathcal{T}X)$, with:

- $(X, \mathrm{P}(X))$ a PCS, and $\mathcal{T}X$ closed by $(\cdot)^{\perp\!\!\perp}$
- $A^{\perp\!\!\perp} = \{u \in \mathbb{R}_{\geq 0}^{|A|} \mid \forall v \in A, \langle u, v\rangle = 1\}$.

Total elements model *terminating* probabilistic computations.

Total elements in Bool.



$|\mathrm{Bool}| = \{\mathtt{t}, \mathtt{f}\}$
$\{\text{elements of } \mathrm{Bool}\} = \{(p, q)\ ;\ p + q \leq 1\}$
$\mathcal{T}\mathrm{Bool} = \{(p, q)\ ;\ p + q = 1\}$

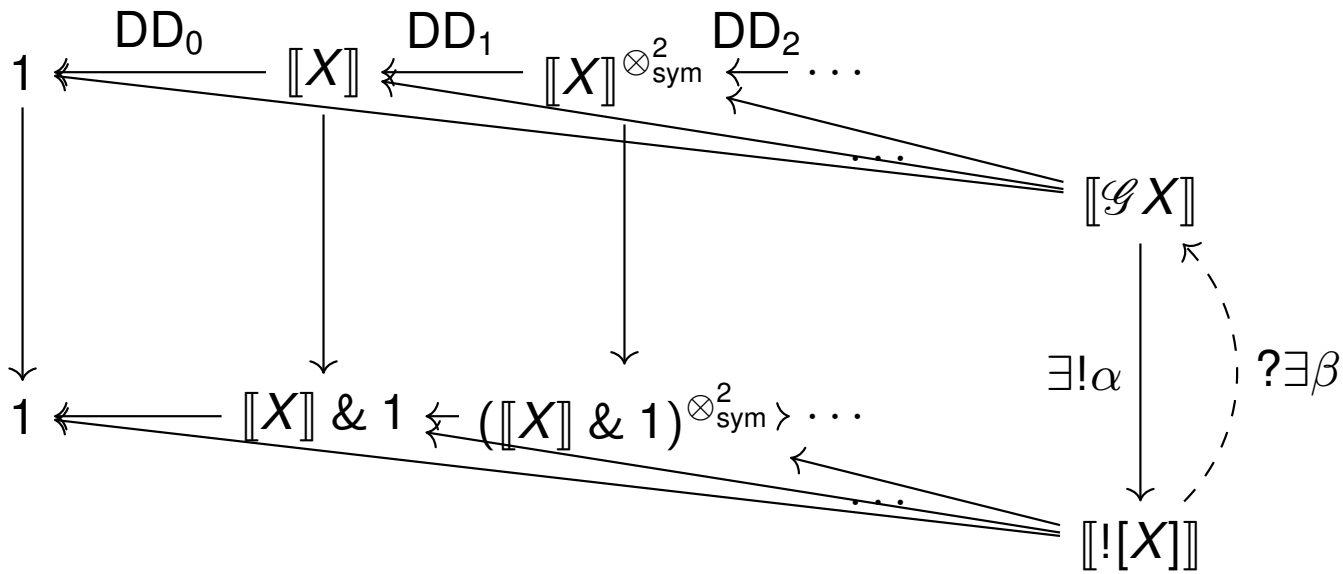
$\Rightarrow$ proper probability distributions on booleans.

Total Elements in $!A$: $\mathcal{T}(!A) = \{x^! \mid x \in \mathcal{T}A\}^{\perp\!\!\perp}$

Total Elements in $!X$

Theorem

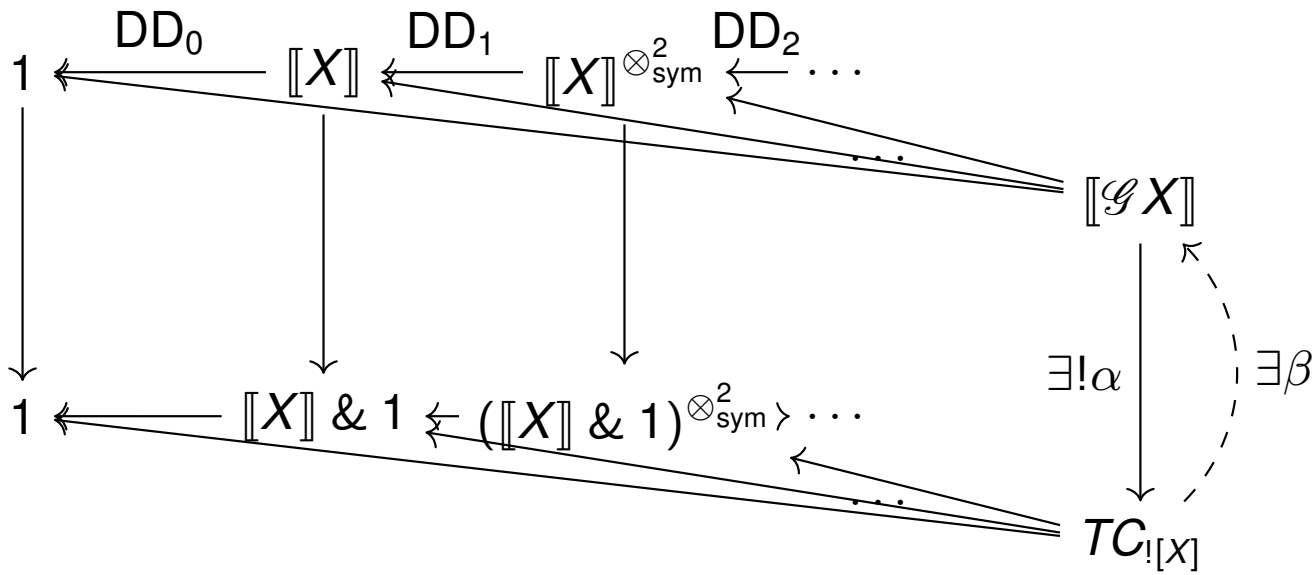
X countable discrete data type. Total elements in $![X]$ are exactly the exchangeable samplers obtained as $\alpha(\eta)$, for $\eta \in [\![\mathscr{G}(\mathscr{G}X)]\!]$.



Total Elements in $!X$

Theorem

X countable discrete data type. Total elements in $![X]$ are exactly the exchangeable samplers obtained as $\alpha(\eta)$, for $\eta \in [\![\mathscr{G}(\mathscr{G}X)]\!]$.



Conclusion

- Exchangeability is at the heart of both the Lafont model of linear logic, and De Finetti theorem.
- In a probabilistic setting $![X]$ can be understood as the set of all exchangeable samplers.
- De Finetti theorem gives us a complete caracterisation of the total elements in $![X]$.

Perspectives

- Does the layered construction on the free exponential hold in ICones ?
- Can we import in semantics other tools from probability theory ?
- Can we exploit the fact that many continuous cones in ICones (e.g. $[\![[0,1]]\!]$) can be expressed as limits of PCoh chains ?