Stable Semantics of Probabilistic Higher-Order Programs

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The Discrete Models of PCSs

Probabilistic Coherence Spaces

Proving operational properties on programs denotationnally.

The Continuous Stable Model

Construction of the Continuous Model: An Overview

Probabilistic Stability

Adding measurability constraints.

Higher-order Probabilistic Programming

Higher-order languages extended with:

- Discrete randomized algorithms (e.g. randomized sorting);
- continuous probability distributions (e.g. to model physical systems);
- Bayesian reasoning (e.g. machine learning algorithms.)

$$
\begin{aligned}
M \in \mathrm{PCF}_{\oplus} ::= \ & x \mid \lambda x^{A} \cdot M \mid (MN) \mid (YN) \\
& \mid \mathrm{ifz}\,(M, N, L) \mid \mathrm{let}(x, M, N) \\
& \mid M \oplus N \mid \underline{n} \mid \mathrm{succ}\,(M) \mid \mathrm{pred}\,(M) \qquad n \in \mathbb{N} \\
& \mid \mathrm{sample} \mid \underline{r} \mid \underline{f} \qquad r \in \mathbb{R}, f : \mathbb{R} \to \mathbb{R} \text{ mesureable} \\
& \mid \mathrm{score} \mid \mathrm{normalize}
\end{aligned}
$$

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Examples

Discrete randomized programs

$M \downarrow \mathscr{D}$ discrete sub-distribution $\quad \mathscr{D} : \{ \text{ normal forms}\} \rightarrow [0,1].$

$$M = \underline{0} \oplus \underline{1} \qquad M \downarrow \frac{1}{2}\{\underline{0}^1\} + \frac{1}{2}\{\underline{1}^1\}$$

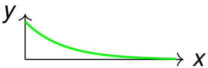
$$N = Y(\lambda x.N \rightarrow N \lambda y.N(y \oplus x(\text{succ}(y))))\underline{0} \qquad N \downarrow \sum_{n \in \mathbb{N}} \frac{1}{2^{n+1}}\{\underline{n}^1\}$$

Program sampling from a continuous distribution

$M \downarrow \mathscr{D}$ continuous sub-distribution $\quad \mathscr{D} \in \Sigma_{\{\text{ normal forms }\}} \rightarrow [0,1].$

- $(\lambda x.\text{sample} + x)1 \downarrow \mu_{[1,2]}$
  with $\mu_{[1,2]}$ Lebesgue Measure on $[1,2]$;
- Exponential Distribution: $M := \text{let}(x, \text{sample}, -\log(x))$

$$M \downarrow (E \mapsto \int_E e^{-x})$$



Density Function

Two stable Models of Higher-Order Probabilistic Computations

PCF + discrete probabilities $\xrightarrow{[\![\cdot]\!]}$ **$\mathbf{PCoh}_!$**

fully abstract

probabilistic coherence spaces (Danos, Ehrhard 2011)

built as a model of LL

PCF + continuous probabilities $\xrightarrow{[\![\cdot]\!]}$ **$\mathbf{Cstab}_{\mathbf{m}}$**

?

measurable complete cones measurable stable functions (Ehrhard, Pagani, Tasson 2018)

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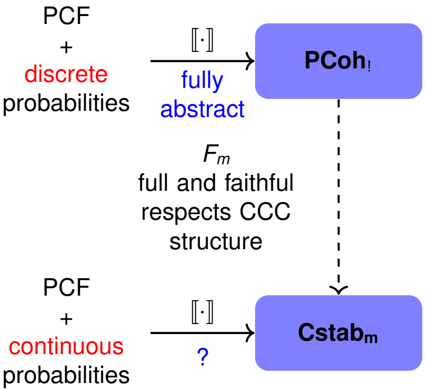

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Outline

1. The Discrete Models of PCSs
   - Probabilistic Coherence Spaces
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2. The Continuous Stable Model
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PCoh: A Linear Logic model for discrete probabilistic computations (Danos, Ehrhard 2011).

Probabilistic coherence spaces (PCS)

pair $(|\mathcal{A}|, \mathrm{P}(\mathcal{A}))$ where:

- $|\mathcal{A}|$: *countable* web;
- $\mathrm{P}(\mathcal{A}) \subseteq \mathbb{R}^{|\mathcal{A}|}$: quantitative cliques.

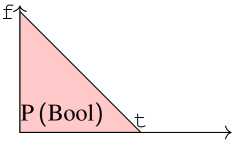
Orthogonality Relation

- For $u, v \in \mathbb{R}^{|\mathcal{A}|}$, $\langle u, v \rangle = \sum_{a \in |\mathcal{A}|} u_a \cdot v_a$;
- $\mathcal{A}^{\perp} = \{u \in \mathbb{R}^{|\mathcal{A}|} \mid \forall v \in \mathrm{P}(\mathcal{A}), \langle u, v \rangle \leq 1\}$.

Bi-Orthogonality Condition on PCSs

$$\mathcal{A} = \mathcal{A}^{\perp\perp}$$

Example (Booleans)



$$|\mathrm{Bool}| = \{\mathtt{t}, \mathtt{f}\}$$
$$\mathrm{P}(\mathrm{Bool}) = \{(p, q) \,;\, p + q \leq 1\} = \{(1,0), (0,1)\}^{\perp\perp}$$

$\Rightarrow$ sub-probability distributions on booleans.

Linear Morphisms in **PCoh**

Morphisms of PCS

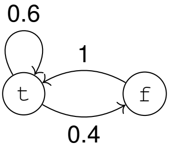
PCoh$(\mathcal{A},\mathcal{B})$: matrices $\phi \in \mathbb{R}^{|\mathcal{A}|} \times |\mathcal{B}|$
$=$ linear functions $\mathbb{R}^{|\mathcal{A}|} \to \mathbb{R}^{|\mathcal{B}|}$.

such that:

$$\forall x \in \mathrm{P}\left(\mathcal{A}\right), \phi(x) \in \mathrm{P}\left(\mathcal{B}\right).$$

Example $(\mathrm{Bool} \multimap \mathrm{Bool})$

$|\mathrm{Bool} \multimap \mathrm{Bool}| = |\mathrm{Bool}| \times |\mathrm{Bool}|$.
$\mathrm{P}\left(\mathrm{Bool} \multimap \mathrm{Bool}\right) = \{u \mid u_{\mathtt{t},\mathtt{t}} + u_{\mathtt{t},\mathtt{f}} \leq 1 \wedge u_{\mathtt{f},\mathtt{t}} + u_{\mathtt{f},\mathtt{f}} \leq 1\}$



$\Rightarrow$ Markov transitions.

PCoh Exponential Modality

Exponential Comonad

- web: $|!\mathcal{A}| = \mathcal{M}_f(|\mathcal{A}|)$
- cliques:
  $\mathrm{P}(!\mathcal{A}) = \{x^! \mid x \in \mathrm{P}(\mathcal{A})\}^{\perp\perp}$

Promotion of x:

$x^! \in \mathbb{R}^{\mathcal{M}_f(|\mathcal{A}|)}$;
$x^!_{[a_1,\ldots,a_n]} := \prod_i x_{a_i}$.

Theorem (C., Ehrhard, Pagani, Tasson)

PCoh *is a Lafont model of Linear Logic*
i.e. $\forall \mathcal{A}$ *PCS,* $!A$ *is the free commutative comonoid generated by* $\mathcal{A}$.

$\mathbf{PCoh}_!$ morphisms

Functional meaning for $f \in \mathbb{R}^{\mathcal{M}_f(|\mathcal{A}|)\times|\mathcal{B}|}$:

$$\widehat{f} : \mathrm{P}(\mathcal{A}) \to (\mathbb{R} \cup +\infty)^{|\mathcal{B}|}$$
$$x \mapsto f \cdot x^!$$

Lemma

$f \in \mathbf{PCoh}_!(\mathcal{A}, \mathcal{B})$ *iff* $\widehat{f}$ *preserves cliques.*

Structure of Functional Interpretations for $\mathbf{PCoh}_!(\mathcal{A}, \mathcal{B})$

Power series from $\mathrm{P}(\mathcal{A})$ to $\mathrm{P}(\mathcal{B})$ with:

- non-negative coefficients;
- a countable number of variables.

$\mathbf{PCoh}_!$ morphisms

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Lemma

$f \in \mathbf{PCoh}_!(\mathcal{A}, \mathcal{B})$ iff $\widehat{f}$ preserves cliques.

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$\mathbf{PCoh}_!$ as model of $\mathrm{PCF}_\oplus$.

Example (M : **Bool** $\rightarrow$ **1**)

$$M = \text{let } b = \textbf{true} \oplus \textbf{false} \text{ in}$$
$$\text{fix } (\lambda f.\lambda x. \text{ if } x = b \text{ then } \star \text{ else } fx).$$

The program M:

- chooses randomly a boolean;
- calls its argument until the result coincides with the boolean chosen.

$$\widehat{[\![M]\!]} : \mathrm{P}\,(\textbf{Bool}) \rightarrow \mathrm{P}\,(\textbf{1}) = [0,1]$$
$$(x_{\mathtt{t}}, x_{\mathtt{f}}) \mapsto \frac{1}{2}\sum_{n\geq 1} x_{\mathtt{f}}^{n-1} \cdot x_{\mathtt{t}} + \frac{1}{2}\sum_{n\geq 1} x_{\mathtt{t}}^{n-1} \cdot x_{\mathtt{f}}$$

Equivalence on Programs

Definition (Context Equivalence)

$M \equiv^{\text{ctx}} N$ when: $\forall$ context $\mathcal{C}$, Obs$(\mathcal{C}[M]) =$ Obs$(\mathcal{C}[N])$.

In a probabilistic setting, *Obs*() can be:

- the probability of termination of a program;
- the probability of returning 0 (for ground types program) . . .

Theorem

- **PCoh**$_!$: *full abstraction for CBN probabilistic PCF. [EPT POPL'2014] i.e.:*
  $M \equiv^{ctx} N \Leftrightarrow [\![M]\!] = [\![N]\!]$.
- **PCoh**$^!$: *full abstraction for a probabilistic version of Levy's CBPV [ET'2016].*

Crucial component in the proof:

A power series is entirely characterized by its coefficients.

Toward a quantitative generalization of context equivalence(1)

From there:

- Can we also express **quantitative properties** in the model ?
- What kind of quantitative operational properties do we want to model ?

Context Distance

A quantitative generalization of Context Equivalence:

$$\delta^{\text{ctx}}(M, N) = \sup_{\mathcal{C} \text{ a context}} |\mathcal{C}[M] - \mathcal{C}[N]|.$$

Problem with Context Distance:

Contexts may be *too powerful*
i.e. amplify too much the distance between programs.

Toward a quantitative generalization of context equivalence(2)

Example (Two very similar programs at context distance 1)

$$M = \textbf{true}$$
$$N_\epsilon = \textbf{true} \oplus^\epsilon \textbf{false} \qquad \text{with } \epsilon \ll 1$$

We can show that $\forall \epsilon > 0$: $\delta^{\text{ctx}}(M, N_\epsilon) = 1$.

Proof.

We can construct a sequence of **amplification contexts** $\mathcal{C}_n$ such that:

$$Obs(\mathcal{C}_n[\textbf{true}]) = 1 \qquad Obs(\mathcal{C}_n[\textbf{true} \oplus^\epsilon \textbf{false}]) = (1-\varepsilon)^n$$

$$\mathcal{C}_n = (\lambda x.\text{if } \underbrace{(x \wedge \ldots \wedge x)} \text{ then } I \text{ else } \Omega)[\cdot].$$

□

Toward a quantitative generalization of context equivalence(3)

Proposition (C., Dal Lago 2017)

In a probabilistic λ-calculus where programs ***almost surely terminate****, all programs are at distance either* 0 *or* 1*.*

Remark

- $\mathrm{PCF}_{\oplus}$ has non-deterministic programs,
- but context distance may nonetheless be a too strong notion.

Tamed Context Distances

Idea

Each time a context uses its argument, it must pay a price in its contribution to context distance.

Definition (Tamed context distances (Ehrhard 2019))

For p a dyadic number in $[0, 1]$:

$$\delta_p^{\text{ctx}}(M, N) = \sup_{\mathcal{C} \text{ a context}} |\mathcal{C}[\Omega \oplus^p M] - \mathcal{C}[\Omega \oplus^p N]|.$$

Theorem (Metric Adequacy of PCSs (Ehrhard 2019))

$$\delta_p^{ctx}(M, N) \leq \frac{p}{1-p} d([\![M]\!], [\![N]\!])$$

where:

- *d is defined using the norm in* **PCoh***;* $\|x\|_{\mathcal{A}} = \sup_{y \in A^{\perp}} \langle x, y \rangle$.
- *For every $t \in \mathrm{P}(\mathcal{A})$, t' represents the derivative of t (seen as a power series).*

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Question:

Can we generalize this fully-abstract model of discrete probabilistic PCF to continuous probabilistic PCF ?

Problem

Distribution on a continuous data-type (e.g. $\mathbb{R}$) cannot be seen as vectors over a web.
$\Rightarrow$Does not allow to model continuous computations.

Stable semantics

A generalization of PCSs to a continuous setting.

Kozen's semantics of First-Order Language with Continuous Probabilities

Kozen's Language

A first-order while language with a random number generator.

Kozen's semantics

- Possible configurations of the memory: **Measurable Spaces**; equipped with a cone structure to manage recursion.
- Program interpretation: **Stochastic Kernels** between measurable spaces.

Stochastic Kernels

$(X, \Sigma_X), (Y, \Sigma_Y)$ two measurable spaces $k : X \times \Sigma_Y \to [0, 1]$ such that:

- $\forall B \in \Sigma_Y$, $(x \in X \mapsto k(x, B))$ is X- measurable.
- $\forall x \in X$, $(B \in \Sigma_Y \mapsto k(x, B))$ is a probability measure on Y.

Semantics for Probabilistic Higher-Order Languages

Fact

Neither **Kern** nor **Meas** are cartesian closed categories.

Staton et al's Quasi Borel spaces

Idea: considering a cartesian closed category of presheaves embedding **Meas**.
$\Rightarrow$ Replace measurable spaces with space of the form $(X, V(X))$, where:

- X is any set
- $V \subseteq (\mathbb{R} \to X)$ is a set of *random variables*.

The measurability constraints on the space are replaced by constraints on the set of random variables.

Example (Quasi-Borel spaces)

A continuous data-type:
The n-uples of reals
$(\mathbb{R}^n, \{f : \mathbb{R} \to \mathbb{R}^n \mid$
f measurable$\})$

A discrete data-type:
The booleans
$(\{0, 1\}, \{f : \mathbb{R} \to \{0, 1\} \mid$
f characteristic function of Borel set$\})$

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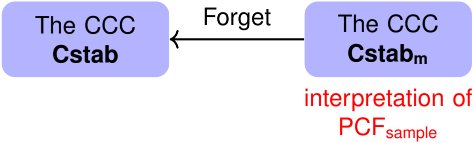
Construction of the Continuous Stable Model[Ehrhard, Pagani, Tasson]

Two steps:

- **Cstab**: The category of complete cones and stable functions on cones;
  $\Rightarrow$ Generalizing PCS to spaces without an underlying countable web
- **Cstab$_\mathbf{m}$**: The category obtained from **Cstab** by adding measurability constraints
  $\Rightarrow$ using random variables in a similar way as in Staton et al's QBS.



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From PCSs to complete cones

Definition (Order on a PCS)

$\mathcal{A} = (|\mathcal{A}|, \mathrm{P}(\mathcal{A}))$ a PCS, $x, y \in \mathrm{P}(\mathcal{A})$:

$$x \leq_{\mathcal{A}} y \quad \text{when} \quad \forall a \in |\mathcal{A}|, x_a \leq y_a.$$

Properties of the order $\leq_{\mathcal{A}}$

- $\mathrm{P}(\mathcal{A})$ is an ω-cpo;
  $\Rightarrow$ allows to interpret fixpoints of programs.
- $\forall x, y \in \mathrm{P}(\mathcal{A})$: $\quad (x \leq_{\mathcal{A}} y \quad \Leftrightarrow \quad \exists z \in \mathrm{P}(\mathcal{A}), y = x + z.)$
  $\Rightarrow$ it gives a cone structure to $\mathrm{P}(\mathcal{A})$.

Illustration on function space

$f, g \in \mathbf{PCoh}_!(\mathcal{A}, \mathcal{B})$ such that $f \leq_{!\mathcal{A} \multimap \mathcal{B}} g$.

- $\widehat{f} \leq \widehat{g}$ coefficient-wise;
- $\widehat{f} - \widehat{g}$ is still a power series with non-negative coefficients.

The CCC $\mathbf{Cstab_m}$ [EPT(2018)] :(1)- Cones

Definition (Cones)

C: $\mathbb{R}$-semimodule, $\|\cdot\|_C : C \to \mathbb{R}$ with equational constraints on $+, \|\cdot\|_C$.

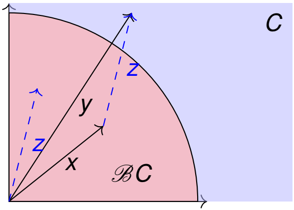
$$(x + y = x + y') \Rightarrow y = y' \qquad \|\alpha x\|_C = \alpha \|x\|_C$$

$$\|x + x'\|_C \leq \|x\|_C + \|x'\|_C \qquad \|x\|_C = 0 \Rightarrow x = 0$$

$$\color{red}{\|x\|_C \leq \|x + x'\|_C}$$

Definitions:

- **Closed Unit Ball** $\mathscr{B}C$;
  $\mathscr{B}C ::= \{x \mid \|x\|_C \leq 1\}$
- **partial order** $\preceq_C$.
  $x \preceq_C y ::= \exists z \in C,\ y = x + z.$
- If $x \preceq_C y$, $\mathbf{x} - \mathbf{y}$ is the *unique* element z of C s.t. $x + z = y$.



Additional Requirement:

Order-Completeness of the Unit Ball

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The CCC **Cstab** [EPT(2018)] :(2)

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Example (Some Complete Cones)

- $\mathbf{1} = (\mathbb{R}, |.|)$
- Non-negative cone of Lebesgues spaces:
  $L_1^+(\mathbb{R}^n) \subseteq \mathbb{R}^n \to \mathbb{R}_+$
  $\|f\| = \int_{x \in \mathbb{R}^n} f \cdot dx < \infty$.
- Meas(X): finite measures over a measurable space X,
  $\|\mu\|_{\text{Meas}(X)} = \mu(X)$.
  $[\![R]\!] = \text{Meas}(\mathbb{R})$

Building a CCC from cones ?

Morphisms in **Cstab**: First Idea

Scott-Continuous Functions $f : \mathscr{B}C \to \mathscr{B}D$.

However (Ehrhard Pagani Tasson)

It does not give a CCC
$\Rightarrow$ We need stronger requirements on morphisms.

Example (Parallel Or)

$$\texttt{or} : \mathscr{B}\mathbf{1} \times \mathscr{B}\mathbf{1} \to \mathscr{B}\mathbf{1} \qquad \texttt{curr(or)} : \mathscr{B}\mathbf{1} \to (\mathscr{B}\mathbf{1} \to \mathscr{B}\mathbf{1})$$
$$x, y \mapsto x + y - xy \qquad x \mapsto (y \mapsto x + y - xy)$$

- `or` is Scott-continuous $\Rightarrow$ it should be a morphism
- For the category to be CCC, it will require `curr(or)` also a morphism.
  It cannot be so: `curr(or)` is not non-decreasing.

Building a CCC from cones ?

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Scott-Continuous Functions $f : \mathscr{B}C \to \mathscr{B}D$.

since

$\texttt{curr(or)}(0) \not\leq_{1\to 1} \texttt{curr(or)}(1)$:

Proof:

$$\texttt{curr(or)}(0) = (y \mapsto y)$$
$$\texttt{curr(or)}(1) = (y \mapsto 1)$$

$\texttt{curr(or)}(1) - \texttt{curr(or)}(0) = 1 - y$

not non-decreasing $\Rightarrow$ not in $\mathbf{1} \to \mathbf{1}$.

H

It

orphisms.

E

$\texttt{r)} : \mathscr{B}\mathbf{1} \to (\mathscr{B}\mathbf{1} \to \mathscr{B}\mathbf{1})$

$x \mapsto (y \mapsto x + y - xy)$

a morphism

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Girard-Berry Stability

Coherent space

$X = (|X|, \mathsf{Cl}(X))$:

- The elements of $\mathsf{Cl}(X)$ are subset of $|X|$, that moreover are cliques of an underlying graph.
- The elements of $\mathsf{Cl}(X)$ are ordered by inclusion.

Definition (Stable function)

X, Y coherent spaces. $f : \mathsf{Cl}(X) \to \mathsf{Cl}(Y)$ is *stable* when:

- f is not decreasing and Scott-continuous
- Stability condition: if $x \cup y \in \mathsf{Cl}(X)$, then $f(x \cap y) = f(x) \cap f(y)$.

Facts

- The interpretation of parallel or is **not** stable;
- A function $\mathsf{Cl}(X) \to \mathsf{Cl}(Y)$ is stable if and only if $\mathrm{Tr}(f)$ is a morphism in $\mathbf{Coh}_!(X, Y)$.

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A reformulation of Girard-Berry's stability [Ehrhard, Pagani, Tasson]

Definition (Local coherent space)

X a coherent space, $u \in \mathrm{Cl}(X)$. X_u: the local PCS at u:

- $|X_u| = \{a \in |X| \mid \{a\} \frown_X u\}$
- $x \frown_{X_u} y$ when $x \frown_X y$.

Proposition (Caracterisation of stable functions)

$f : \mathrm{P}(X) \to \mathrm{P}(Y)$ is stable if and only if:

$$\forall u \in Cl(X),\ \Delta f_u \text{ is non-decreasing},$$

where Δf_u is the local difference of f at u:

$$\begin{aligned} \Delta f_u : & Cl(E_u) \to Cl(F) \\ & v \mapsto f(u \cup v) - f(v). \end{aligned}$$

Pre-Stable Functions

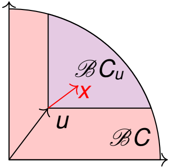
Definition (n-pre-stable functions for $n \in \mathbb{N}$)

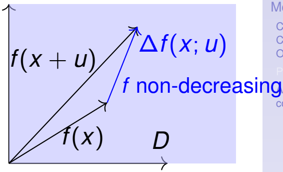
$f : \mathscr{B}C \to \mathscr{B}D$ with:

- $n = 0$: f is non-decreasing
- $n > 0$: $\forall u \in \mathscr{B}C$, $\Delta f_{|u}$ is $(n-1)$ pre-stable.

Notations:

- Local Cone at u.
- Local Differences.





$\Delta f_{|u} : x \in \mathscr{B}C_u \mapsto \Delta f(x;u)$

Example (3-pre-stability)

- $f(x) \preceq_D f(x+u)$;
- $f(x+u_1) + f(x+u_2) \preceq_D f(x+u_1+u_2) + f(x)$;
- $f(x+u_1+u_2) + f(x+u_1+u_3) + f(x+u_2+u_3) + f(x) \preceq_D f(x+u_1+u_2+u_3) + f(x+u_1) + f(x+u_2) + f(x+u_3)$.

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Proving operational properties on programs denotationnally.
The Continuous Stable Model
Construction of the Continuous Model: An Overview
Probabilistic Stability
Adding measurability constraints.

The CCC **Cstab**.

Example (∞ pre-stable functions)

- *linear* functions;
- every class of functions preserved by $f \mapsto \Delta f_{|u}$:
  e.g. in $\mathbb{R}^n \to \mathbb{R}$:
  - polynomials with non-negative coefficients
  - power series with non-negative coefficients

Definition (Probabilistic Stable functions $\mathscr{B}C \to \mathscr{B}D$)

- Scott-continuous.
- ∞ pre-stable

Definition (The CCC **Cstab**)

- Objects : complete cones
- Morphisms: stable functions.

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Definition (Probabilistic Stable functions $\mathscr{B}C \to \mathscr{B}D$)

- Scott-continuous.
- ∞ pre-stable $\Rightarrow$ excludes the Scott-continuous function denoting the parallel or

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Connection between **PCoh$_!$** and **Cstab.**

Theorem (C 2018)

There exists a functor $F : \mathbf{PCoh}_! \to \mathbf{Cstab_m}$:

- *which is full and faithful*
- *and respects the cartesian closed structure.*

Proof sketch

- Every PCS can be seen as a complete cone;
- Stable functions on PCSs coincide with power series with non-negative coefficients:
  - uses a result due to McMillan on absolutely monotonous functions on partitions systems
  - from a stable function, build *generalized derivatives* that allows to recover the power series coefficients.

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Why do we need additional constraints on stable functions ?

Goal:

Sample first, then do a computation.

Example

$M := \texttt{let}\, x = L \,\texttt{in}\, N.$

- $x : \mathbb{R} \vdash N : \mathbb{R} \quad \Rightarrow \quad [\![N]\!] : \text{Meas}(\mathbb{R}) \mapsto \text{Meas}(\mathbb{R}).$
- $\vdash L : \mathbb{R} \quad \Rightarrow \quad [\![L]\!] : \text{Meas}(\mathbb{R})$

We would like the interpretation of M to be:

$$[\![M]\!] \in [\![\mathbb{R}]\!] = \text{Meas}(\mathbb{R})$$
$$= U \in \Sigma_{\mathbb{R}} \mapsto \int f.d[\![L]\!]$$

where $f : r \in \mathbb{R} \mapsto [\![N]\!](\delta_r)(U) \in \mathbb{R}.$

$\Rightarrow$ We need to add constraints to guarantee that this integral makes sense

Measurable Cones

Definition (Mesureable cone)

C equipped of a family of random variables $\text{Paths}^n(C) \subseteq \mathbb{R}^n \to C$, such that:

- For every $\gamma \in \text{Paths}^n(C)$, $\gamma(\mathbb{R}^n)$ is bounded in C;
- $\forall \gamma \in \text{Paths}^n(C)$, $f : \mathbb{R}^m \mapsto \mathbb{R}^n$ measurable, $\gamma \circ f \in \text{Paths}^m(C)$;
- $\forall x \in C$, $n \in \mathbb{N}$, $(\vec{r} \in \mathbb{R}^n \mapsto x) \in \text{Paths}^n(C)$.

Example (The Cone $\text{Meas}(X)$ of Bounded Measures on X)

Unitary measurable paths: Stochastic Kernels from $\mathbb{R}$ into the measurable space X.

Definition (Measurable Functions $f : \mathscr{B}C \to D$.)

$\forall \gamma \in \text{Paths}^n(C)$ with $\gamma(\mathbb{R}^n) \subseteq \mathscr{B}C$, $f \circ \gamma \in \text{Paths}^n(D)$.

i.e. it preserves measurable paths.

Stable Semantics

The $\mathbf{Cstab_m}$ category

- Objects: Measurable Cones
- Morphisms: stable measurable functions $\mathscr{B}C \to \mathscr{B}D$.

Theorem

Ehrhard, Pagani, Tasson $\mathbf{Cstab_m}$ *is an adequate model of* PCF_{sample}.

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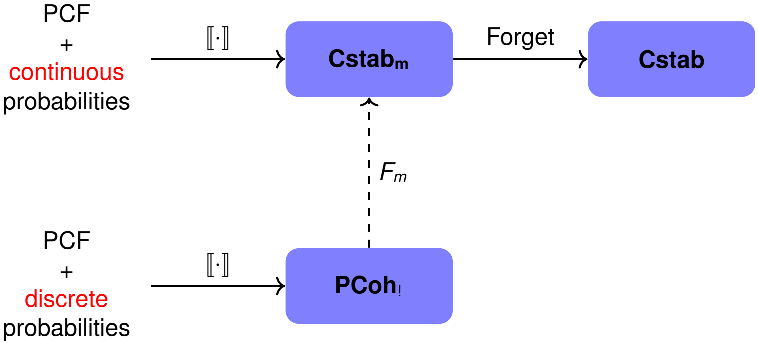
$\mathbf{Cstab_m}$ is a conservative extension of $\mathbf{PCoh_!}$.



Theorem (C 2018)

We can extend F into a functor $F_m : \mathbf{PCoh_!} \to \mathbf{Cstab_m}$ that

- *is full and faithful,*
- *preserves the cartesian closed structure.*

Conclusion

Probabilistic coherence space models

- full abstraction results
- Express quantitative property on programs (Ehrhard tamed distance)
- genericity of its exponential structure (i.e. it is a Lafont model)

Stable continuous model

A generalization of PCSs where coexist:

- Wild data structures (e.g. bounded measures over any measurable set)
- Very regular morphisms (that can be understood using our result as generalization of analytic functions).

Perspectives

- Find a model of linear logic with $\mathbf{Cstab_m}$ as Kleisli category.
- Extension of $\mathbf{PCoh}_!$ full abstraction proof for $\mathrm{PCF}_{\oplus}$ to $\mathbf{Cstab_m}$ for $\mathrm{PCF}_{\mathrm{sample}}$.

Stable Semantics of Probabilistic Higher-Order Programs
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$$C \xrightarrow{f \text{ stable function}} \mathbf{1}$$

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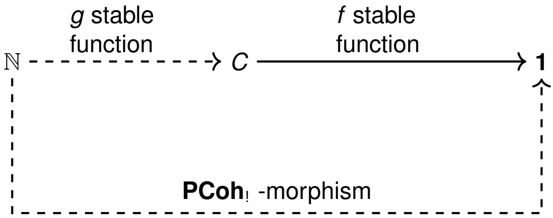

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