

# On Probabilistic Applicative Bisimulation and Call-by-Value $\lambda$ -Calculi (Long Version)

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## Abstract

Probabilistic applicative bisimulation is a recently introduced coinductive methodology for program equivalence in a probabilistic, higher-order, setting. In this paper, the technique is generalized to a typed, call-by-value, lambda-calculus. Surprisingly, the obtained relation coincides with context equivalence, contrary to what happens when call-by-name evaluation is considered. Even more surprisingly, full-abstraction only holds in a symmetric setting.

## 1 Introduction

Traditionally, an algorithm is nothing but a finite description of a sequence of deterministic primitive instructions, which solve a computational problem when executed. Along the years, however, this concept has been generalized so as to reflect a broader class of effective procedures and machines. One of the many ways this has been done consists in allowing probabilistic choice as a primitive instruction in algorithms, this way shifting from usual, deterministic computation to a new paradigm, called probabilistic computation. Examples of application areas in which probabilistic computation has proved to be useful include natural language processing [20], robotics [29], computer vision [3], and machine learning [23]. Sometimes, being able to “flip a fair coin” while computing is a *necessity* rather than an alternative, like in computational cryptography (where, e.g., secure public key encryption schemes must be probabilistic [11]).

Any (probabilistic) algorithm can be executed by concrete machines only once it takes the form of a *program*. And indeed, various probabilistic programming languages have been introduced in the last years, from abstract ones [16, 27, 22] to more concrete ones [24, 12]. A quite common scheme consists in endowing any deterministic language with one or more primitives for probabilistic choice, like binary probabilistic choice or primitives for distributions.

Viewing algorithms as functions allows a smooth integration of distributions into the playground, itself nicely reflected at the level of types through monads [13, 27]. As a matter of fact, some existing probabilistic programming languages [24, 12] are designed around the  $\lambda$ -calculus or one of its incarnations, like *Scheme*. This, in turn has stimulated foundational research about probabilistic  $\lambda$ -calculi, and in particular about the nature of program equivalence in a probabilistic setting. This has already started to produce some interesting results in the realm of denotational semantics, where adequacy and full-abstraction results have recently appeared [7, 9].

Not much is known about operational techniques for probabilistic program equivalence, and in particular about coinductive methodologies. This is in contrast with what happens for deterministic or nondeterministic programs, when various notions of bisimulation have been introduced and proved to be adequate and, in some cases, fully abstract. A recent paper by Alberti, Sangiorgi and the second author [10] generalizes Abramsky’s applicative bisimulation [1] to  $\Lambda_{\oplus}$ , a call-by-name, untyped  $\lambda$ -calculus endowed with binary, fair, probabilistic choice [6]. Probabilistic applicative bisimulation is shown to be a congruence, thus included in context equivalence. Completeness, however, fails, the counterexample being exactly the one separating bisimulation and context

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equivalence in a nondeterministic setting. Full abstraction is then recovered when pure, deterministic  $\lambda$ -terms are considered, as well as well another, more involved, notion of bisimulation, called coupled logical bisimulation, takes the place of applicative bisimulation.

In this paper, we proceed with the study of probabilistic applicative bisimulation, analysing its behaviour when instantiated on call-by-value  $\lambda$ -calculi. This investigation brings up some nice, unexpected results. Indeed, not only the non-trivial proof of congruence for applicative bisimulation can be adapted to the call-by-value setting, which is somehow expected, but applicative bisimilarity turns out to precisely characterize context equivalence. This is quite surprising, given that in nondeterministic  $\lambda$ -calculi, both when call-by-name and call-by-value evaluation are considered, applicative bisimilarity is a congruence, but *finer* than context equivalence. There is another, even less expected result: the aforementioned correspondence does not hold anymore if we consider applicative *simulation* and the contextual *preorder*.

Technically, the presented results owe much to a recent series of studies about probabilistic bisimulation for labelled Markov processes [8, 30], i.e., labelled probabilistic transition systems in which the state space is continuous (rather than discrete, as in Larsen and Skou’s labelled Markov chains [18]), but time stays discrete. More specifically, the way we prove that context equivalent terms are bisimilar goes by constructively show how each *test* of a kind characterizing probabilistic bisimulation can be turned into an equivalent *context*. If, as a consequence, two terms are not bisimilar, then any test the two terms satisfy with different probabilities (of which there must be at least one) becomes a context in which the two terms converges with different probabilities. This also helps understanding the discrepancies between the probabilistic and nondeterministic settings, since in the latter the class of tests characterizing applicative bisimulation is well-known to be quite large [21].

The whole development is done in a probabilistic variation on PCF with lazy lists, called  $\text{PCFL}_{\oplus}$ : working on an applied calculus allows to stay closer to concrete programming languages, this way facilitating exemplification, as in Section 2 below.

## 2 Some Motivating Examples

In this section, we want to show how  $\lambda$ -calculus can express interesting, although simple, probabilistic programs. More importantly, we will argue that checking the equivalence of some of the presented programs is not only interesting from a purely theoretical perspective, but corresponds to a proof of *perfect security* in the sense of Shannon [28].

Let’s start from the following very simple programs:

$$\begin{aligned} NOT &= \lambda x. \text{if } x \text{ then } \underline{\text{false}} \text{ else } \underline{\text{true}} : \mathbf{bool} \rightarrow \mathbf{bool}; \\ ENC &= \lambda x. \lambda y. \text{if } x \text{ then } (NOT \ y) \text{ else } y : \mathbf{bool} \rightarrow \mathbf{bool} \rightarrow \mathbf{bool}; \\ GEN &= \underline{\text{true}} \oplus \underline{\text{false}} : \mathbf{bool}. \end{aligned}$$

The function  $ENC$  computes exclusive disjunction as a boolean function, but can also be seen as the encryption function of a one-bit version of the so-called One-Time Pad cryptoscheme (OTP in the following). On the other hand,  $GEN$  is a term reducing probabilistically to one of the two possible boolean values, each with probability  $\frac{1}{2}$ , and is meant to be a way to generate a random key for the same scheme.

One of the many ways to define perfect security of an encryption scheme consists in setting up an *experiment* [17]: the adversary generates two messages, of which one is randomly chosen, encrypted, and given back to the adversary who, however, should not be able to guess whether the first or the second message have been chosen (with success probability strictly greater than  $\frac{1}{2}$ ). This can be seen as the problem of proving the following two programs to be context equivalent:

$$\begin{aligned} EXP &= \lambda x. \lambda y. ENC \ (x \oplus y) \ GEN : \mathbf{bool} \rightarrow \mathbf{bool} \rightarrow \mathbf{bool}; \\ RND &= \lambda x. \lambda y. \underline{\text{true}} \oplus \underline{\text{false}} : \mathbf{bool} \rightarrow \mathbf{bool} \rightarrow \mathbf{bool}. \end{aligned}$$

where  $\oplus$  is a primitive for fair, probabilistic choice. Analogously, one could verify that any adversary is not able to distinguish an experiment in which the *first* message is chosen from an experiment in which the *second* message is chosen. This, again, can be seen as the task of checking whether the following two terms are context equivalent:

$$\begin{aligned} EXP_{FST} &= \lambda x. \lambda y. ENC\ x\ GEN : \mathbf{bool} \rightarrow \mathbf{bool} \rightarrow \mathbf{bool}; \\ EXP_{SND} &= \lambda x. \lambda y. ENC\ y\ GEN : \mathbf{bool} \rightarrow \mathbf{bool} \rightarrow \mathbf{bool}. \end{aligned}$$

But how could we actually *prove* context equivalence? The universal quantification in its definition, as is well known, turns out to be burdensome in proofs. The task can be made easier by way of various techniques, including context lemmas and logical relations. Later in this paper, we show how the four terms above can be shown equivalent by way of applicative bisimulation, which is proved sound (and complete) with respect to context equivalence in Section 4 below.

Before proceeding, we would like to give examples of terms having the same type, but which are *not* context equivalent. We will do so by again referring to perfect security. The kind of security offered by the OTP is unsatisfactory not only because keys cannot be shorter than messages, but also because it does not hold in presence of multiple encryptions, or when the adversary is *active*, for example by having an access to an encryption oracle. In the aforementioned scenario, security holds if and only if the following two programs (both of type  $\mathbf{bool} \rightarrow \mathbf{bool} \rightarrow \mathbf{bool} \times (\mathbf{bool} \rightarrow \mathbf{bool})$ ) are context equivalent:

$$\begin{aligned} EXP_{FST}^{CPA} &= \lambda x. \lambda y. (\lambda z. \langle ENC\ x\ z, \lambda w. ENC\ w\ z \rangle) GEN; \\ EXP_{SND}^{CPA} &= \lambda x. \lambda y. (\lambda z. \langle ENC\ y\ z, \lambda w. ENC\ w\ z \rangle) GEN. \end{aligned}$$

It is very easy, however, to realize that if  $C = (\lambda x. (\text{snd}(x))(\text{fst}(x)))([\cdot]\ \text{true}\ \text{false})$ , then  $C[EXP_{FST}^{CPA}]$  reduces to  $\text{true}$ , while  $C[EXP_{SND}^{CPA}]$  reduces to  $\text{false}$ , both with probability 1. In other words, the OTP is not secure in presence of active adversaries, and for very good reasons: having access to an oracle for encryption is essentially equivalent to having access to an oracle for *decryption*.

### 3 Programs and Their Operational Semantics

In this section, we will present the syntax and operational semantics of  $\text{PCFL}_{\oplus}$ , the language on which we will define applicative bisimulation. The language  $\text{PCFL}_{\oplus}$  is identical to Pitts'  $\text{PCFL}$  [25], except for the presence of a primitive for binary probabilistic choice.

#### 3.1 Terms and Types

The terms of  $\text{PCFL}_{\oplus}$  are built up from constants (for boolean and integer values, and for the empty list) and variables, using the usual constructs from PCF, and binary choice. In the following,  $\mathcal{V} = \{x, y, \dots\}$  is a countable set of variables and  $\mathcal{O}$  is a finite set of binary arithmetic operators including at least the symbols  $+$ ,  $\leq$ , and  $=$ .

**Definition 1** Terms are expressions generated by the following grammar:

$$\begin{aligned} M, N ::= & x \mid \underline{n} \mid \underline{b} \mid \text{nil} \mid \langle M, M \rangle \mid M :: M \mid \lambda x. M \mid \text{fix } x. M \\ & \mid M \oplus M \mid \text{if } M \text{ then } M \text{ else } M \mid M \text{ op } M \mid \text{fst}(M) \mid \text{snd}(M) \\ & \mid \text{case } M \text{ of } \{\text{nil} \rightarrow M \mid h :: t \rightarrow M\} \mid M M, \end{aligned}$$

where  $x, h, t \in \mathcal{V}$ ,  $n \in \mathbb{N}$ ,  $b \in \mathbb{B} = \{\text{true}, \text{false}\}$ ,  $\text{op} \in \mathcal{O}$ .

In what follows, we consider terms of  $\text{PCFL}_{\oplus}$  as  $\alpha$ -equivalence classes of syntax trees. The set of free variables of a term  $M$  is indicated as  $FV(M)$ . A term  $M$  is closed if  $FV(M) = \emptyset$ . The (capture-avoiding) substitution of  $N$  for the free occurrences of  $x$  in  $M$  is denoted  $M[N/x]$ .

The constructions from PCF have their usual meanings. The operator  $(\cdot :: \cdot)$  is the constructor for lists,  $\text{nil}$  is the empty list, and  $\text{case } L \text{ of } \{\text{nil} \rightarrow M \mid h :: t \rightarrow N\}$  is a list destructor. The construct  $M \oplus N$  is a binary choice operator, to be interpreted probabilistically, as in  $\Lambda_{\oplus}$  [6].

**Example 1** *Relevant examples of terms are  $\Omega = (\text{fix } x. x) \underline{0}$ , and  $I = \lambda x. x$ : the first one always diverges, while the second always converges (to itself). In between, one can find terms that converge with probability between 0 and 1, excluded, e.g.,  $I \oplus \Omega$ , and  $I \oplus (I \oplus \Omega)$ .*

We are only interested in well-formed terms, i.e., terms to which one can assign a type.

**Definition 2** *Types are given by the following grammar:*

$$\begin{aligned} \sigma, \tau &::= \gamma \mid \sigma \rightarrow \sigma \mid \sigma \times \sigma \mid [\sigma]; \\ \gamma, \delta &::= \mathbf{bool} \mid \mathbf{int}. \end{aligned}$$

The set of all types is  $\mathcal{Y}$ . Please observe that the language of types we consider here coincides with the one of Pitts' PCFL [25]. An alternative typing discipline for probabilistic languages (see, e.g. [27]), views probability as a *monad*, this way reflecting the behaviour of programs in types: if  $\sigma$  is a type,  $\square\sigma$  is the type of probabilistic distributions over  $\sigma$ , and the binary choice operator always produces elements of type  $\square\sigma$ .

**Example 2** *The following expressions are types:  $\mathbf{int}$ ,  $\mathbf{int} \times \mathbf{bool}$ ,  $\mathbf{int} \rightarrow (\mathbf{bool} \times \mathbf{int})$ .*

We assume that all operators from  $\mathcal{O}$  take natural numbers as input, and we associate to each operator  $\text{op} \in \mathcal{O}$  its *result type*  $\gamma_{\text{op}} \in \{\mathbf{bool}, \mathbf{int}\}$  and its semantics  $\overline{\text{op}} : \mathbb{N} \times \mathbb{N} \rightarrow X$  where  $X$  is either  $\mathbb{B}$  or  $\mathbb{N}$ , depending on  $\gamma_{\text{op}}$ . A *typing context*  $\Gamma$  is a finite partial function from variables to types.  $\text{dom}(\Gamma)$  is the domain of the function  $\Gamma$ . If  $x \notin \text{dom}(\Gamma)$ ,  $(x : \sigma, \Gamma)$  represents the function which extends  $\Gamma$  to  $\text{dom}(\Gamma) \cup \{x\}$ , by associating  $\sigma$  to  $x$ .

**Definition 3** *A typing judgement is an assertion of the form  $\Gamma \vdash M : \sigma$ , where  $\Gamma$  is a context,  $M$  is a term, and  $\sigma$  is a type. A judgement is valid if it can be derived by the rules of the formal system given in Figure 1.*

$\frac{(x, \sigma) \in \Gamma}{\Gamma \vdash x : \sigma}$	$\frac{b \in \mathbb{B}}{\Gamma \vdash \underline{b} : \mathbf{bool}}$	$\frac{n \in \mathbb{N}}{\Gamma \vdash \underline{n} : \mathbf{int}}$	$\frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \sigma \quad \Gamma \vdash L : \mathbf{bool}}{\Gamma \vdash \text{if } L \text{ then } M \text{ else } N}$
$\frac{\Gamma \vdash M : \mathbf{int} \quad \Gamma \vdash N : \mathbf{int}}{\Gamma \vdash M \text{ op } N : \gamma_{\text{op}}}$	$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x. M : \sigma \rightarrow \tau}$	$\frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M \oplus N : \sigma}$	
$\frac{\Gamma \vdash M : \sigma \rightarrow \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}$	$\frac{\Gamma, x : \sigma \rightarrow \tau \vdash M : \sigma \rightarrow \tau}{\Gamma \vdash \text{fix } x. M : \sigma \rightarrow \tau}$	$\frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \tau}{\Gamma \vdash \langle M, N \rangle : \sigma \times \tau}$	
$\frac{\Gamma \vdash M : \sigma \times \tau}{\Gamma \vdash \text{fst}(M) : \sigma}$	$\frac{\Gamma \vdash M : \sigma \times \tau}{\Gamma \vdash \text{snd}(M) : \tau}$	$\frac{}{\Gamma \vdash \text{nil} : [\sigma]}$	$\frac{\Gamma \vdash T : [\sigma] \quad \Gamma \vdash H : \sigma}{\Gamma \vdash H :: T : [\sigma]}$
$\frac{\Gamma \vdash L : [\sigma] \quad \Gamma \vdash M_1 : \tau \quad \Gamma, h : \sigma, t : [\sigma] \vdash M_2 : \tau}{\Gamma \vdash \text{case } L \text{ of } \{\text{nil} \rightarrow M_1 \mid h :: t \rightarrow M_2\} : \tau}$			

Figure 1: Type Assignment in  $\text{PCFL}_{\oplus}$

Please notice that any term of which we want to form the fixpoint needs to be a function.

**Definition 4** *If  $\sigma$  is a type and  $\Gamma$  is a typing context, then  $\mathcal{T}^{\sigma} = \{t \mid \emptyset \vdash t : \sigma\}$ ,  $\mathcal{T} = \{t \mid \exists \sigma, t \in \mathcal{T}^{\sigma}\}$ ,  $\mathcal{T}_{\Gamma}^{\sigma} = \{t \mid \Gamma \vdash t : \sigma\}$ .*

In other words,  $\mathcal{T}^\sigma$  is the set of *closed terms* (also called *programs*) of type  $\sigma$ , while  $\mathcal{T}$  is the set of closed terms which have a valid typing derivation, and  $\mathcal{T}_\Gamma^\sigma$  is the set of terms which have type  $\sigma$  under the context  $\Gamma$ . We can observe that  $\mathcal{T}^\sigma = \mathcal{T}_\emptyset^\sigma$ .

**Example 3** *The following type assignments are valid:*

- $\forall \Gamma$  a context , and  $\sigma$  a type:  $\Gamma \vdash I : \sigma \rightarrow \sigma$ ;
- For every function type  $\tau$ , and all typing context  $\Gamma$ ,  $\Gamma \vdash \text{fix } x. x : \tau$ ;
- The previous point allow us to see that for all type  $\sigma$ ,  $\emptyset \vdash \text{fix } x. x : \mathbf{int} \rightarrow \sigma$ . So  $\forall \Gamma$  a context , and  $\sigma$  a type:  $\Gamma \vdash \Omega : \sigma$ ;
- $\forall \Gamma$  a context , and  $\sigma$  a type:  $\Gamma \vdash I \oplus \Omega : \sigma \rightarrow \sigma$ ;
- $\emptyset \vdash \text{fix } x. ((\lambda z. \underline{0}) \oplus \lambda z. ((x \underline{0}) + \underline{1})) : \mathbf{int} \rightarrow \mathbf{int}$ .

### 3.2 Operational Semantics

Because of the probabilistic nature of choice in  $\text{PCFL}_\oplus$ , a program doesn't evaluate to a value, but to a probability distribution of values. Therefore, we need the following notions to define an evaluation relation.

**Definition 5** *Values are terms of the following form:*

$$V ::= \underline{n} \mid \underline{b} \mid \text{nil} \mid \lambda x. M \mid \text{fix } x. M \mid M :: M \mid \langle M, M \rangle.$$

We will call  $\mathcal{V}$  the set of values, and we note  $\mathcal{V}^\sigma = \mathcal{V} \cap \mathcal{T}^\sigma$ . A value distribution is a function  $\mathcal{D} : \mathcal{V} \rightarrow [0, 1]$ , such that  $\sum_{V \in \mathcal{V}} \mathcal{D}(V) \leq 1$ . Given a value distribution  $\mathcal{D}$ , we will note  $\mathcal{S}(\mathcal{D})$  the set of those values  $V$  such that  $\mathcal{D}(V) > 0$ . A value distribution  $\mathcal{D}$  is said *finite* whenever  $\mathcal{S}(\mathcal{D})$  has finite cardinality. If  $V$  is a value, we note  $\{V^1\}$  the value distribution  $\mathcal{D}$  such that  $\mathcal{D}(W) = 1$  if  $W = V$  and  $\mathcal{D}(V) = 0$  otherwise. Value distributions can be ordered pointwise.

We first give an *approximation* semantics, which attributes *finite* probability distributions to terms, and only later define the actual semantics, which will be the least upper bound of all distributions obtained through the approximation semantics. Big-step semantics is given by way of a binary relation  $\Downarrow$  between closed terms and value distributions, which is defined by the set of rules from Figure 2. This evaluation relation, by the way, is the natural extension to  $\text{PCFL}_\oplus$  of the evaluation relation given in [6] for the untyped probabilistic  $\lambda$ -calculus. Please observe how function arguments are evaluated before being passed to functions. Moreover,  $M :: N$  is a value even if  $M$  or  $N$  are not, which means that lists are *lazy* and potentially infinite.

**Proposition 1** *Call-by-value evaluation preserves typing, that is: if  $M \Downarrow \mathcal{D}$ , and  $M \in \mathcal{T}^\sigma$ , then for every  $V \in \mathcal{S}(\mathcal{D})$ ,  $V \in \mathcal{V}^\sigma$ .*

**Lemma 1** *For every term  $M$ , if  $M \Downarrow \mathcal{D}$ , and  $M \Downarrow \mathcal{E}$ , then there exists a distribution  $\mathcal{F}$  such that  $M \Downarrow \mathcal{F}$  with  $\mathcal{D} \leq \mathcal{F}$ , and  $\mathcal{E} \leq \mathcal{F}$ .*

**Proof.** The proof is by induction on the structure of derivations for  $M \Downarrow \mathcal{D}$ . □

**Definition 6** *For any closed term  $M$ , we define the big-steps semantics  $\llbracket M \rrbracket$  of  $M$  as  $\sup_{M \Downarrow \mathcal{D}} \mathcal{D}$ .*

Since distributions form an  $\omega$ -complete partial order, and for every  $M$  the set of those distributions  $\mathcal{D}$  such that  $M \Downarrow \mathcal{D}$  is a countable directed set (by Lemma 1), this definition is well-posed, and associates a unique value distribution to every term. In [6] various ways to define coinductively call-by-value approximation semantics on probabilistic untyped  $\lambda$ -calculus were introduced, and it was proved that the semantics obtained by taking the greatest lower bound of this coinductive approximation semantics was equivalent to the inductively characterized semantics. Although it is possible to extend similarly those definitions for  $\text{PCFL}_\oplus$  we do not do it, and only limit our attention to inductively defined probabilistic semantics.

The distribution  $\llbracket M \rrbracket$  can be obtained equivalently by taking the least upper bound of all finite distributions  $\mathcal{D}$  for which  $M \Rightarrow \mathcal{D}$ , where  $\Rightarrow$  is a binary relation capturing *small-step* evaluation

$$\begin{array}{c}
\frac{}{M \Downarrow \emptyset} b_e \quad \frac{}{V \Downarrow \{V^1\}} b_v \quad \frac{M \Downarrow \mathcal{D} \quad N \Downarrow \mathcal{E}}{M \text{ op } N \Downarrow \sum_{n \in S(\mathcal{D}), m \in S(\mathcal{E})} \mathcal{D}(n) \mathcal{E}(m) \{\underline{n \text{ op } m}^1\}} b_{op} \\
\\
\frac{M \Downarrow \mathcal{K} \quad N \Downarrow \mathcal{F} \quad \{P[v/x] \Downarrow \mathcal{E}_{P,v}\}_{\lambda x. P \in S(\mathcal{X}), v \in S(\mathcal{F})} \quad \{Q[\text{fix } x. Q/x]v \Downarrow \mathcal{G}_{Q,v}\}_{\text{fix } x. Q \in S(\mathcal{X}), v \in S(\mathcal{F})}}{MN \Downarrow \sum_{v \in S(\mathcal{F})} \mathcal{F}(v) \left( \sum_{\lambda x. P \in S(\mathcal{X})} \mathcal{K}(\lambda x. P) \cdot \mathcal{E}_{P,v} + \sum_{\text{fix } x. Q \in S(\mathcal{X})} \mathcal{K}(\text{fix } x. Q) \cdot \mathcal{G}_{Q,v} \right)} b_a \\
\\
\frac{L \Downarrow \mathcal{D} \quad M_1 \Downarrow \mathcal{E}_1 \quad M_2 \Downarrow \mathcal{E}_2}{\text{if } L \text{ then } M_1 \text{ else } M_2 \Downarrow \mathcal{D}(\text{true})\mathcal{E}_1 + \mathcal{D}(\text{false})\mathcal{E}_2} b_{if} \\
\\
\frac{L \Downarrow \mathcal{D} \quad M_1 \Downarrow \mathcal{E} \quad (H \Downarrow \mathcal{G}_{(H,T)}, T \Downarrow \mathcal{K}_{(H,T)})_{H::T \in S(\mathcal{D})} \quad \{M_2\{V/h, W/t\} \Downarrow \mathcal{F}_{V,W}\}_{H::T \in S(\mathcal{D}), V \in S(\mathcal{G}_{(H,T)}), W \in S(\mathcal{K}_{(H,T)})}}{\text{case } L \text{ of } \{\text{nil} \rightarrow M_1 \mid h :: t \rightarrow M_2\} \Downarrow \mathcal{D}(\text{nil})\mathcal{E} + \sum_{H::T \in S(\mathcal{D})} \sum_{\substack{V \in S(\mathcal{G}_{(H,T)}) \\ W \in S(\mathcal{K}_{(H,T)})}} \mathcal{D}(H :: T) \mathcal{G}_{(H,T)}(V) \mathcal{K}_{(H,T)}(W) \mathcal{F}_{V,W}} b_{case} \\
\\
\frac{M \Downarrow \mathcal{D} \quad \{P \Downarrow \mathcal{E}_P\}_{(P,N) \in S(\mathcal{D})}}{\text{fst}(M) \Downarrow \sum_{(P,N) \in S(\mathcal{D})} \mathcal{D}(\langle P, N \rangle) \cdot \mathcal{E}_P} b_{fst} \quad \frac{M \Downarrow \mathcal{D} \quad \{N \Downarrow \mathcal{E}_N\}_{(P,N) \in S(\mathcal{D})}}{\text{snd}(M) \Downarrow \sum_{(P,N) \in S(\mathcal{D})} \mathcal{D}(\langle P, N \rangle) \cdot \mathcal{E}_N} b_{snd} \\
\\
\frac{M_1 \Downarrow \mathcal{D}_1 \quad M_2 \Downarrow \mathcal{D}_2}{M_1 \oplus M_2 \Downarrow \frac{1}{2} \mathcal{D}_1 + \frac{1}{2} \mathcal{D}_2} b_s
\end{array}$$

Figure 2: Evaluation — Rule Selection

of terms. This proceeds as follows. The first step consists in defining the notion of an *evaluation context*, which in the case of  $\text{PCFL}_{\oplus}$  is the following one

$$\begin{array}{l}
E ::= [\cdot] \mid EM \mid VE \mid E \text{ op } M \mid V \text{ op } E \mid \mid \text{fst}(E) \mid \text{snd}(E) \mid \text{if } E \text{ then } M \text{ else } M \\
\mid \text{case } E \text{ of } \{\text{nil} \rightarrow M \mid h :: t \rightarrow M\} \mid \text{case } E :: M \text{ of } \{\text{nil} \rightarrow M \mid h :: t \rightarrow M\} \\
\mid \text{case } V :: E \text{ of } \{\text{nil} \rightarrow M \mid h :: t \rightarrow M\}
\end{array}$$

The next step consists in giving a relation modelling one-step reduction. In our step this takes the form of a relation  $\rightarrow$  between closed terms and *sequences* of closed terms, which is defined as the smallest such relation including satisfying the following rules:

$$\begin{array}{l}
(\lambda x. M)V \rightarrow N[V/x]; \\
(\text{fix } x. M)V \rightarrow (M[(\text{fix } x. M)/x])V; \\
\underline{n \text{ op } m} \rightarrow \underline{\text{op}(m, n)}; \\
\text{fst}(\langle V, W \rangle) \rightarrow V; \\
\text{snd}(\langle V, W \rangle) \rightarrow W; \\
\text{if true then } M \text{ else } N \rightarrow M; \\
\text{if false then } M \text{ else } N \rightarrow N; \\
\text{case nil of } \{\text{nil} \rightarrow M \mid h :: t \rightarrow N\} \rightarrow M; \\
\text{case } V :: W \text{ of } \{\text{nil} \rightarrow M \mid h :: t \rightarrow N\} \rightarrow N[V, W/h, t].
\end{array}$$

and closed under all evaluation contexts, i.e., if  $M \rightarrow N_1, \dots, N_n$ , then we also have that  $E[M] \rightarrow E[N_1], \dots, E[N_n]$ . Proper probabilistic computation enters the playground as soon as we define the relation  $\Rightarrow$  between closed terms and value distributions, which is defined by inductively

interpreting the following three rules:

$$\frac{}{M \Rightarrow \emptyset} \quad \frac{}{V \Rightarrow \{V^1\}} \quad \frac{M \rightarrow N_1, \dots, N_n \quad N_i \Rightarrow \mathcal{D}_i}{M \Rightarrow \sum_{1 \leq i \leq n} \frac{1}{n} \mathcal{D}_i}$$

**Theorem 1 (Big-step is Equivalent to Small-step [6])**  $\llbracket M \rrbracket = \sup_{M \Rightarrow \mathcal{D}} \mathcal{D}$ .

**Example 4** *Approximation semantics does not allow to derive any assertion about  $\Omega$ , and indeed  $\llbracket \Omega \rrbracket = \emptyset$ . Similarly,  $\llbracket I \rrbracket = \{I^1\}$ . Recursion allows to define much more interesting programs, e.g.  $M = (\text{fix } x. (\lambda y. y) \oplus \lambda y. x(y + \underline{1})) \underline{0}$ . Indeed,  $\llbracket M \rrbracket(\underline{n}) = \frac{1}{2^{n+1}}$  for every  $n \in \mathbb{N}$ , even if  $M \not\Downarrow \llbracket M \rrbracket$ . As another example,  $\llbracket (\lambda x. I \oplus \lambda x. \Omega) \underline{0} \rrbracket = \frac{1}{2} \{I^1\}$ . Finally,  $\llbracket (\text{fix } x. I \oplus x) \underline{0} \rrbracket = \{\underline{0}^1\}$ , but please observe that we don't have  $(\text{fix } x. (I \oplus x)) \underline{0} \Downarrow \{\underline{0}^1\}$ .*

### 3.3 Relations

The notion of *typed relation* corresponds to a family of relations  $(\mathcal{R}_\sigma^\Gamma)_{\sigma, \Gamma}$ , each of them a binary relation on  $\mathcal{T}_\sigma^\Gamma$ . We extend the usual notion of symmetry, reflexivity and transitivity to typed relations in the following way:

**Definition 7** *A typed relation is a family  $\mathcal{R} = (\mathcal{R}_\sigma^\Gamma)_{\sigma, \Gamma}$ , where each  $\mathcal{R}_\sigma^\Gamma$  is a binary relation on  $\mathcal{T}_\sigma^\Gamma$ . Sometime,  $M \mathcal{R}_\sigma^\Gamma N$  will be noted as  $\Gamma \vdash M \mathcal{R}_\sigma N$  (or as  $\Gamma \vdash M \mathcal{R} N : \sigma$ ). A typed relation  $\mathcal{R}$  is said to be:*

- reflexive if  $\forall M \in \mathcal{T}_\sigma^\Gamma$  it holds that  $\Gamma \vdash M \mathcal{R} M : \sigma$ ;
- symmetric if  $\forall \sigma, \Gamma. \forall M, N \in \mathcal{T}_\sigma^\Gamma. \Gamma \vdash M \mathcal{R} N : \sigma \Rightarrow \Gamma \vdash N \mathcal{R} M : \sigma$ ;
- transitive if  $\forall \sigma, \Gamma. \forall M, N, L \in \mathcal{T}_\sigma^\Gamma. (\Gamma \vdash M \mathcal{R} N : \sigma \wedge \Gamma \vdash N \mathcal{R} L : \sigma) \Rightarrow \Gamma \vdash M \mathcal{R} L : \sigma$ .

**Definition 8** *Let  $\mathcal{R}$  be a typed relation. We define the compatibility of  $\mathcal{R}$  in the expected way. For instance, if  $\mathcal{R}$  is compatible, the following properties should hold:*

- $\Gamma \vdash \underline{k} \mathcal{R} \underline{k} : \text{int}$  for every  $k \in \mathbb{N}$ ;
- $x : \tau, \Gamma \vdash x \mathcal{R} x : \tau$  for every  $x$  and for every  $\tau$ ;
- $\Gamma \vdash M \mathcal{R} N : \sigma$  and  $\Gamma \vdash L \mathcal{R} P : \sigma$  implies  $\Gamma \vdash (M \oplus L) \mathcal{R} (N \oplus P) : \sigma$ .

Please observe that a compatible typed relation  $\mathcal{R}$  is always reflexive, since  $\mathcal{R}$  is reflexive for terms of ground form, and that  $\mathcal{R}$  is stable by the constructors of the language:

**Proposition 2** *Let  $\mathcal{R}$  be a typed relation. If  $\mathcal{R}$  is compatible, then  $\mathcal{R}$  is reflexive.*

Any typed relation capturing a notion of equivalence should be a congruence, this way being applicable at any point in the program, possibly many times:

**Definition 9** *Let  $\mathcal{R}$  be a typed relation. Then  $\mathcal{R}$  is said to be a precongruence relation if  $\mathcal{R}$  is transitive and compatible, and  $\mathcal{R}$  is said to be a congruence relation if  $\mathcal{R}$  is symmetric, transitive and compatible.*

We write  $\mathcal{R}$  for the set of type-indexed families  $\mathcal{R} = (\mathcal{R}_\sigma)_\sigma$  of binary relations  $\mathcal{R}_\sigma$  between the terms in  $\mathcal{T}^\sigma$ .

### 3.4 Context Equivalence

The general idea of context equivalence is the following: two terms  $M$  and  $N$  are equivalent if any occurrence of  $M$  in any program  $L$  can be replaced with  $N$  without changing the observable behaviour of  $L$ . The notion of a context allows us to formalize this idea.

**Definition 10** A context is a syntax tree with a unique hole:

$$\begin{aligned}
C ::= & [\cdot] \mid \lambda x.C \mid CM \mid MC \mid C \oplus M \mid M \oplus C \\
& \mid \text{fix } x.C \mid M \text{ op } C \mid C \text{ op } M \mid \langle C, M \rangle \mid \langle M, C \rangle \mid \text{fst}(C) \mid \text{snd}(C) \\
& \mid \text{if } M \text{ then } M \text{ else } C \mid \text{if } M \text{ then } C \text{ else } M \mid \text{if } C \text{ then } M \text{ else } M \mid M :: C \mid C :: M \\
& \mid \text{case } C \text{ of } \{\text{nil} \rightarrow M \mid h :: t \rightarrow M\} \mid \text{case } M \text{ of } \{\text{nil} \rightarrow M \mid h :: t \rightarrow C\} \mid \text{case } M \text{ of } \{\text{nil} \rightarrow C \mid h :: t \rightarrow M\}
\end{aligned}$$

Given a context  $C$  and a term  $M$ ,  $C[M]$  is the term obtained by substituting the unique hole in  $C$  with  $M$ .

When defining context equivalence, we work with *closing* contexts, namely those contexts  $C$  such that  $C[M]$ , and  $C[N]$  are closed terms (where  $M$  and  $N$  are the possibly open terms being compared). We are now going to define a notion of typing for contexts. Judgments have the shape  $\Gamma \vdash C(\Delta; \sigma) : \tau$ , which informally means that if  $M$  is a term of type  $\sigma$  under the typing context  $\Delta$ , then the hole of  $C$  can be filled by  $M$ , obtaining a term of type  $\tau$  in the context  $\Gamma$ . So in order to define well-typed substitutions for a term in a context, we extend, in a natural way, the notion of typing to contexts:

**Definition 11** A typing judgement for contexts is an assertion of the form:  $\Gamma \vdash C(\Delta; \sigma) : \tau$ , where  $\Gamma, \Delta$  are typing contexts,  $C$  is a context, and  $A, B$  are types. A judgement is valid if it can be derived by the rules of the formal system given in Figure 3.

The operation  $M \mapsto C[M]$  of substituting a  $\text{PCFL}_{\oplus}$  term for a parameter in a context to obtain a new  $\text{PCFL}_{\oplus}$  term respect typing in the following sense:

**Proposition 3** Let be  $\Gamma, \Delta$ , such that  $\text{dom}(\Gamma) \cap \text{dom}(\Delta) = \emptyset$ . Let be  $M$  such that  $\Gamma, \Delta \vdash M : \sigma$ , and  $C$  a context such that:  $\Gamma \vdash C(\Delta; \sigma) : \tau$ . Then  $\Gamma \vdash C[M] : \tau$ .

**Proof.** The proof is by induction on the structure of the derivation of  $\Gamma \vdash C(\Delta, \sigma) : \tau$ . □

**Example 5** Example of derivable judgments of the just described form are  $\emptyset \vdash \lambda x.[\cdot](x : \sigma; \tau) : (\sigma \rightarrow \tau)$  and  $\emptyset \vdash ((\lambda x.\text{true})[\cdot]) (\emptyset; \sigma) : \mathbf{bool}$ .

Here, following [7, 10, 9], we consider that the observable behaviour of a program  $M$  is its *probability of convergence*  $\sum \llbracket M \rrbracket = \sum_V \llbracket M \rrbracket(V)$ . We now have all the ingredients necessary to define what context equivalence is:

**Definition 12** The contextual preorder is the typed relation  $\leq$  given by: for every  $M, N \in \mathcal{T}_{\tau}^{\Gamma}$ ,  $\Gamma \vdash M \leq N : \tau$  if for every context  $C$  such that  $\emptyset \vdash C(\Gamma; \tau) : \sigma$ , it holds that  $\sum \llbracket C[M] \rrbracket \leq \sum \llbracket C[N] \rrbracket$ . Context equivalence is the typed relation  $\equiv$  given by stipulating that  $\Gamma \vdash M \equiv N : \sigma$  iff  $\Gamma \vdash M \leq N : \sigma$  and  $\Gamma \vdash N \leq M : \sigma$ .

Another way to define context equivalence would be to restrain ourselves to contexts of **bool** and **int** type in the definition of context equivalence: this is the so-called *ground* context equivalence. In a call-by-value setting, however, this gives exactly the same relation, since any non-ground context can be turned into a ground context inducing the same probability of convergence. A similar argument holds for a notion of equivalence in which one observes the obtained (ground) *distribution* rather than merely its sum. The following can be proved in a standard way:

**Proposition 4**  $\leq$  is a typed relation, which is reflexive, transitive and compatible.

Because of the quantification over all contexts, it is usually difficult to show that  $M$  and  $N$  are two context equivalent terms. In the next sections, we will introduce another notion of equivalence, and we show that it is included in context equivalence.



$$\begin{array}{c}
\frac{dom(\Gamma) \cap dom(\Delta) = \emptyset}{\Gamma \vdash [\cdot](\Delta; A) : A} \\
\\
\frac{\Gamma, x : \sigma \vdash C(\Delta; B) : \tau \quad x \notin dom(\Gamma), x \notin dom(\Delta)}{\Gamma \vdash \lambda x. C(x : \sigma, \Delta; B) : \sigma \rightarrow \tau} \\
\\
\frac{\Gamma, x : \sigma \vdash C(\Delta; B) : \tau}{\Gamma \vdash \text{fix } x. C(x : \tau, \Delta; B) : \tau} \textit{fix} \\
\\
\frac{\Gamma \vdash C(\Delta; B) : \sigma \rightarrow \tau \quad \Gamma \vdash M : \sigma}{\Gamma \vdash CM(\Delta; B) : \tau} \qquad \frac{\Gamma \vdash C(\Delta; B) : \sigma \quad \Gamma \vdash M : \sigma \rightarrow \tau}{\Gamma \vdash MC(\Delta; B) : \tau} \\
\\
\frac{\Gamma \vdash C(\Delta; B) : \sigma \times \tau}{\Gamma \vdash \text{fst}(C)(\Delta; B) : \sigma} \qquad \frac{\Gamma \vdash C(\Delta; B) : \sigma \times \tau}{\Gamma \vdash \text{snd}(C)(\Delta; B) : \tau} \\
\\
\frac{\Gamma \vdash C(\Delta; B) : \sigma \quad \Gamma \vdash M : \tau}{\Gamma \vdash \langle C, M \rangle(\Delta; B) : \sigma \times \tau} \qquad \frac{\Gamma \vdash C(\Delta; B) : \sigma \quad \Gamma \vdash M : \tau}{\Gamma \vdash \langle M, C \rangle(\Delta; B) : \tau \times \sigma} \\
\\
\frac{\Gamma \vdash C(\Delta; B) : \sigma \quad \Gamma \vdash M : [\sigma]}{\Gamma \vdash C :: M(\Delta; B) : [\sigma]} \qquad \frac{\Gamma \vdash C(\Delta; B) : [\sigma] \quad \Gamma \vdash M : \sigma}{\Gamma \vdash M :: C(\Delta; B) : [\sigma]} \\
\\
\frac{\Gamma \vdash C(\Delta; B) : \sigma \quad \Gamma \vdash M : \sigma}{\Gamma \vdash M \oplus C(\Delta; B) : \sigma} \qquad \frac{\Gamma \vdash C(\Delta; B) : \sigma \quad \Gamma \vdash M : \sigma}{\Gamma \vdash C \oplus M(\Delta; B) : \sigma} \\
\\
\frac{\Gamma \vdash C(\Delta; B) : \mathbf{int} \quad \Gamma \vdash M : \mathbf{int}}{\Gamma \vdash M \text{op} C(\Delta; B) : \gamma_{\text{op}}} \qquad \frac{\Gamma \vdash C(\Delta; B) : \mathbf{int} \quad \Gamma \vdash M : \mathbf{int}}{\Gamma \vdash C \text{op} M(\Delta; B) : \gamma_{\text{op}}} \\
\\
\frac{\Gamma \vdash C(\Delta; B) : \mathbf{bool} \quad \Gamma \vdash M_1 : \sigma \quad \Gamma \vdash M_2 : \sigma}{\Gamma \vdash \text{if } C \text{ then } M_1 \text{ else } M_2(\Delta; B) : \sigma} \\
\\
\frac{\Gamma \vdash C(\Delta; B) : \sigma \quad \Gamma \vdash L : \mathbf{bool} \quad \Gamma \vdash M : \sigma}{\Gamma \vdash \text{if } L \text{ then } C \text{ else } M(\Delta; B) : \sigma} \\
\\
\frac{\Gamma \vdash C(\Delta; B) : \sigma \quad \Gamma \vdash L : \mathbf{bool} \quad \Gamma \vdash M : \sigma}{\Gamma \vdash \text{if } L \text{ then } M \text{ else } C(\Delta; B) : \sigma} \\
\\
\frac{\Gamma \vdash C(\Delta; B) : [\sigma] \quad \Gamma \vdash M_1 : \tau \quad \Gamma, h : \sigma, t : [\sigma] \vdash M_2 : \tau \quad h, t \notin dom(\Gamma)}{\Gamma \vdash \text{case } C \text{ of } \{\text{nil} \rightarrow M_1 \mid h :: t \rightarrow M_2\}(\Delta; B) : \tau} \\
\\
\frac{\Gamma \vdash C(\Delta; B) : \tau \quad \Gamma \vdash M_1 : [\sigma] \quad \Gamma, h : \sigma, t : [\sigma] \vdash M_2 : \tau \quad h, t \notin dom(\Gamma)}{\Gamma \vdash \text{case } M_1 \text{ of } \{\text{nil} \rightarrow C \mid h :: t \rightarrow M_2\}(\Delta; B) : \tau} \\
\\
\frac{\Gamma, h : \sigma, t : [\sigma] \vdash C(\Delta; B) : \tau \quad \Gamma \vdash M_1 : [\sigma] \quad \Gamma \vdash M_2 : \tau \quad h, t \notin dom(\Gamma) \cup dom(\Delta)}{\Gamma \vdash \text{case } M_1 \text{ of } \{\text{nil} \rightarrow M_2 \mid h :: t \rightarrow C\}(\Delta, h : \sigma, t : [\sigma]; B) : \tau}
\end{array}$$

Figure 3: Context Type Assignment

## 4 Applicative Bisimulation

In this section, we introduce the notions of similarity and bisimilarity for  $\text{PCFL}_{\oplus}$ . We proceed by instantiating probabilistic bisimulation as developed by Larsen and Skou for a generic labelled Markov chain in [18]. A similar use was done for a call-by-name untyped probabilistic  $\lambda$ -calculus  $\Lambda_{\oplus}$  in [10].

### 4.1 Larsen and Skou's Probabilistic Bisimulation

Preliminary to the notion of (bi)simulation, is the notion of a *labelled Markov chain* (LMC in the following), which is a triple  $\mathcal{M} = (\mathcal{S}, \mathcal{L}, \mathcal{P})$ , where  $\mathcal{S}$  is a countable set of *states*,  $\mathcal{L}$  is a set of *labels*, and  $\mathcal{P}$  is a *transition probability matrix*, i.e., a function  $\mathcal{P} : \mathcal{S} \times \mathcal{L} \times \mathcal{S} \rightarrow \mathbb{R}$  such that for every state  $s \in \mathcal{S}$  and for every label  $l \in \mathcal{L}$ ,  $\sum_{t \in \mathcal{S}} \mathcal{P}(s, l, t) \leq 1$ . Following [8], we allow the sum above to be smaller than 1, modelling divergence this way. The following is due to Larsen and Skou [18]:

**Definition 13** *Given  $(\mathcal{S}, \mathcal{L}, \mathcal{P})$  a labelled Markov Chain, a probabilistic simulation is a pre-order relation  $R$  on  $\mathcal{S}$  such that  $(s, t) \in R$  implies that for every  $X \subseteq \mathcal{S}$  and for every  $l \in \mathcal{L}$ ,  $\mathcal{P}(s, l, X) \leq \mathcal{P}(t, l, R(X))$ , with  $R(X) = \{y \mid \exists x \in X \text{ such that } x R y\}$ . Similarly, a probabilistic bisimulation is an equivalence relation  $R$  on  $\mathcal{S}$  such that  $(s, t) \in R$  implies that for every equivalence class  $E$  modulo  $R$ , and for every  $l \in \mathcal{L}$ ,  $\mathcal{P}(s, l, E) = \mathcal{P}(t, l, E)$ .*

Insisting on bisimulations to be equivalence relations has the potential effect of not allowing them to be formed by just taking unions of other bisimulations. The same can be said about simulations, which are assumed to be partial orders. Nevertheless:

**Proposition 5** *If  $(R_i)_{i \in I}$  is a collection of probabilistic (bi)simulations, then the reflexive and transitive closure of their union,  $(\cup_{i \in I} R_i)^*$ , is a (bi)simulation.*

A nice consequence of the result above is that we can define *probabilistic similarity* (noted  $\lesssim$ ) simply as the relation  $\lesssim = \bigcup \{R \mid R \text{ is a probabilistic simulation}\}$ . Analogously for the largest probabilistic bisimulation, that we call *probabilistic bisimilarity* (noted  $\smile$ ), defined as  $\smile = \bigcup \{R \mid R \text{ is a probabilistic bisimulation}\}$ .

**Proposition 6** *Any symmetric probabilistic simulation is a probabilistic bisimulation.*

A property of probabilistic bisimulation which does not hold in the usual, nondeterministic, setting, is the following:

**Proposition 7**  $\smile = \lesssim \cap \lesssim^{op}$ .

### 4.2 A Concrete Labelled Markov Chain

Applicative bisimulation will be defined by instantiating Definition 13 on a specific LMC, namely the one modelling evaluation of  $\text{PCFL}_{\oplus}$  programs.

**Definition 14** *The labelled Markov Chain  $\mathcal{M}_{\oplus} = (\mathcal{S}_{\oplus}, \mathcal{L}_{\oplus}, \mathcal{P}_{\oplus})$  is given by:*

- A set of states  $\mathcal{S}_{\oplus}$  defined as follows:

$$\mathcal{S}_{\oplus} = \{(M, \sigma) \mid M \in \mathcal{T}^{\sigma}\} \uplus \{(\hat{V}, \sigma) \mid V \in \mathcal{V}^{\sigma}\},$$

where terms and values are taken modulo  $\alpha$ -equivalence. A value  $V$  in the second component of  $\mathcal{S}_{\oplus}$  is distinguished from one in the first by using the notation  $\hat{V}$ .

- A set of labels  $\mathcal{L}_{\oplus}$  defined as follows:

$$\mathcal{V} \uplus \mathcal{Y} \uplus \mathbb{N} \uplus \mathbb{B} \uplus \{\text{nil}, \text{hd}, \text{tl}\} \uplus \{\text{fst}, \text{snd}\} \uplus \{\text{eval}\},$$

where, again, terms are taken modulo  $\alpha$ -equivalence, and  $\mathcal{Y}$  is the set of types.

- A transition probability matrix  $\mathcal{P}_{\oplus}$  such that:

- For every  $M \in \mathcal{T}^\sigma$ ,  $\mathcal{P}_\oplus((M, \sigma), \sigma, (M, \sigma)) = 1$ , and similarly for values.
- For every  $M \in \mathcal{T}^\sigma$ , and any value  $V \in \mathcal{S}(\llbracket M \rrbracket)$ ,  $\mathcal{P}_\oplus((M, \sigma), eval, (\hat{V}, \sigma)) = \llbracket M \rrbracket(V)$ .
- If  $V \in \mathcal{V}^\sigma$ , then:
  - If  $\sigma = \tau \rightarrow \theta$ , then
    - Either there is  $M$  such that  $V = \lambda x.M$ , and for each  $W \in \mathcal{V}^\tau$ ,

$$\mathcal{P}_\oplus((\hat{V}, \tau \rightarrow \theta), W, (M[W/x], \theta)) = 1.$$

- Or there is  $M$  such that  $V = \text{fix } x.M$ , and for each  $W \in \mathcal{V}^\tau$ ,

$$\mathcal{P}_\oplus((\hat{V}, \tau \rightarrow \theta), W, (M[\text{fix } x.M/x]W, \theta)) = 1.$$

- If  $\sigma = \tau \times \theta$ , then there are  $M, N$  such that  $V = \langle M, N \rangle$ , and we define:

$$\mathcal{P}_\oplus((\hat{V}, \tau \times \theta), fst, (M, \tau)) = 1$$

$$\mathcal{P}_\oplus((\hat{V}, \tau \times \theta), snd, (N, \theta)) = 1$$

- If  $\sigma = \mathbf{int}$ , then there is  $k \in \mathbb{N}$  such that  $V = \underline{k}$  and  $\mathcal{P}_\oplus((\hat{V}, \mathbf{int}), k, (\hat{V}, \mathbf{int})) = 1$ .
- If  $\sigma = \mathbf{bool}$ , then there is  $b \in \mathbb{B}$  such that  $V = \underline{b}$ . Then  $\mathcal{P}_\oplus((\hat{V}, \mathbf{bool}), b, (\hat{V}, \mathbf{bool})) = 1$ .
- If  $\sigma = [\tau]$ , then there are two possible cases:
  - If  $V = \text{nil}$ , then  $\mathcal{P}_\oplus((\hat{V}, [\tau]), \text{nil}, (\hat{V}, [\tau])) = 1$ .
  - If  $V = M :: N$ , then  $\mathcal{P}_\oplus((\hat{V}, [\tau]), hd, (M, \tau)) = 1$ . and  $\mathcal{P}_\oplus((\hat{V}, [\tau]), tl, (N, [\tau])) = 1$ .

For all  $s, l, t$  such that  $\mathcal{P}_\oplus(s, l, t)$  isn't defined above, we have  $\mathcal{P}_\oplus(s, l, t) = 0$ .

Please observe that if  $V \in \mathcal{V}^\sigma$ , both  $(V, \sigma)$  and  $(\hat{V}, \sigma)$  are states of the Markov Chain  $\mathcal{M}_\oplus$ . For example, the following are all states of  $\mathcal{M}_\oplus$ :

$$\begin{aligned} &(\lambda x.x, (\mathbf{int} \rightarrow \mathbf{int})); \\ &(\widehat{\lambda x.x}, (\mathbf{int} \rightarrow \mathbf{int})); \\ &(\lambda x.x, ((\mathbf{int} \rightarrow \mathbf{int}) \rightarrow (\mathbf{int} \rightarrow \mathbf{int}))); \\ &(\widehat{\lambda x.x}, ((\mathbf{int} \rightarrow \mathbf{int}) \rightarrow (\mathbf{int} \rightarrow \mathbf{int}))). \end{aligned}$$

A similar Markov Chain was used in [10] to define bisimilarity for the untyped probabilistic  $\lambda$ -calculus  $\Lambda_\oplus$ . We use here in the same way actions which apply a term to a value, and an action which models term evaluation, namely *eval*.

### 4.3 The Definition

We would like to see any simulation (or bisimulation) on the LMC  $\mathcal{M}_\oplus$  as a family in  $\mathcal{R}$ . As can be easily realized, indeed, any (bi)simulation on  $\mathcal{M}_\oplus$  cannot put in correspondence states  $(M, \sigma)$  and  $(N, \tau)$  where  $\sigma \neq \tau$ , since each such pair exposes its second component as an action. This then justifies the following:

**Definition 15** A probabilistic applicative simulation (a PAS in the following), is a family  $(\mathcal{R}_\sigma) \in \mathcal{R}$  such that there exists a probabilistic simulation  $R$  on the LMC  $\mathcal{M}_\oplus$  such that for every type  $\sigma$ , and for every  $M, N \in \mathcal{T}^\sigma$  it holds that  $M \mathcal{R}_\sigma N \Leftrightarrow (M, \sigma) R (N, \sigma)$ . A probabilistic applicative bisimulation (PAB in the following) is defined similarly, requiring  $R$  to be a bisimulation rather than a simulation.

The greatest simulation and the greatest bisimulation on  $\mathcal{M}_\oplus$  are indicated with  $\lesssim$ , and  $\sim$ , respectively. In other words,  $\lesssim_\sigma$  is the relation  $\{(M, N) \mid (M, \sigma) \lesssim (N, \sigma)\}$ , while  $\sim_\sigma$  the relation  $\{(M, N) \mid (M, \sigma) \sim (N, \sigma)\}$ .

Please notice that  $(\lesssim_\sigma)$  is the biggest PAS, and that  $(\sim_\sigma)$  is the biggest PAB. We can also see that if  $(\mathcal{R}_\sigma)$  is a PAS, and we define the relation  $R$  by: if  $M\mathcal{R}_\sigma N$  then  $(M, \sigma)R(N, \sigma)$ , and if  $V, W$  are values, and  $V\mathcal{R}_\sigma W$ , then  $(\hat{V}, \sigma)R(\hat{W}, \sigma)$ , then  $\mathcal{R}$  is a simulation on  $\mathcal{M}_\oplus$ . Similarly if we start from an PAB.

**Lemma 2** *For every  $(V, W) \in \mathcal{V}^\sigma \times \mathcal{V}^\sigma$ ,  $(\hat{V}, \sigma) \lesssim (\hat{W}, \sigma)$  if and only if  $(V, \sigma) \lesssim (W, \sigma)$ .*

**Proof.**  $\Leftarrow$  If  $(V, \sigma) \lesssim (W, \sigma)$ , we have:

$$\mathcal{P}_\oplus((V, \sigma), eval, X) = \llbracket V \rrbracket(X) = \begin{cases} 1 & \text{if } (\hat{V}, \sigma) \in X \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathcal{P}_\oplus((W, \sigma), eval, (\lesssim(X))) = \begin{cases} 1 & \text{if } (\hat{W}, \sigma) \in \lesssim(X) \\ 0 & \text{otherwise} \end{cases}$$

As  $\lesssim$  is a simulation,  $\mathcal{P}_\oplus((V, \sigma), eval, X) \leq \mathcal{P}_\oplus((W, \sigma), eval, (\lesssim(X)))$ . We take  $X = \{(\hat{V}, \sigma)\}$ , and we can see that we must have  $(\hat{W}, \sigma) \in \lesssim(X)$ , and it follows that  $(\hat{V}, \sigma) \lesssim (\hat{W}, \sigma)$ .  
 $\Rightarrow$  Let  $\sigma$  be a fixed type. Let be  $R = \{((V, \sigma), (W, \sigma)) \mid ((\hat{V}, \sigma) \lesssim (\hat{W}, \sigma))\}$ . We are going to show:  $R \subseteq \lesssim$ . Let  $P = \lesssim \cup R$ . We can see that  $P$  is a simulation: Let  $s, t$  be such that  $sPt$ . Then

- Either  $s \lesssim t$ , and for every action  $l$  and subset  $X$  of  $\mathcal{S}_\oplus$ ,  $\mathcal{P}_\oplus(s, l, X) \leq \mathcal{P}_\oplus(t, l, (\lesssim(X))) \leq \mathcal{P}_\oplus(t, l, (P(X)))$ .
- Or there exist  $V$  and  $W$  such that  $s = (V, \sigma)$ ,  $t = (W, \sigma)$ , and  $(\hat{V}, \sigma) \lesssim (\hat{W}, \sigma)$ , and so  $V, W \in \mathcal{V}^\sigma$  and, for every action  $l$ :
  - either  $l = eval$ , and for every  $X \subseteq \mathcal{S}_\oplus$ ,

$$\mathcal{P}_\oplus(s, eval, X) = \begin{cases} 1 & \text{if } (\hat{V}, \sigma) \in X \\ 0 & \text{otherwise} \end{cases}$$

If  $(\hat{V}, \sigma) \notin X$ ,  $\mathcal{P}_\oplus(s, eval, X) = 0 \leq \mathcal{P}_\oplus(t, eval, (P(X)))$ . If  $(\hat{V}, \sigma) \in X$ , then  $(\hat{W}, \sigma) \in \lesssim(X) \subseteq P(X)$  and so  $\mathcal{P}_\oplus(s, eval, X) = 1 = \mathcal{P}_\oplus(t, eval, (P(X)))$ .

- either  $l \neq eval$ , and for every subset  $X$  of  $\mathcal{S}_\oplus$ :  $\mathcal{P}_\oplus(s, l, X) = \mathcal{P}_\oplus(t, l, P(X)) = 0$

Since  $P$  is a simulation,  $P \subseteq \lesssim$ , and so we have  $\{(V, W) \mid \hat{V} \lesssim \hat{W}\} \subseteq \lesssim$ .

This concludes the proof.  $\square$

Terms having the same semantics need to be bisimilar:

**Lemma 3** *Let  $(\mathcal{R}_\sigma) \in \mathcal{R}$  be defined as follows:  $M \mathcal{R}_\sigma N \Leftrightarrow M, N \in \mathcal{T}^\sigma \wedge \llbracket M \rrbracket = \llbracket N \rrbracket$ . Then  $(\mathcal{R}_\sigma)$  is a PAB.*

**Proof.** Let  $R = \bigcup_\sigma \left( \{((M, \sigma), (N, \sigma)) \mid M\mathcal{R}_\sigma N\} \cup \{((\hat{V}, \sigma), (\hat{V}, \sigma)) \mid V \in \mathcal{V}^\sigma\} \right)$ . We proceed by showing that  $R$  is a bisimulation. Now:

- For every  $\sigma$ ,  $\mathcal{R}_\sigma$  is an equivalence relation, so  $R$  is an equivalence relation too. The equivalence classes of  $R$  are: the  $(\{\sigma\} \times E_\sigma)$  when  $E_\sigma$  is an equivalence class of  $\mathcal{R}_\sigma$ , and the  $\{(\hat{V}, \sigma)\}$  when  $V \in \mathcal{V}^\sigma$ .
- For every  $s, t \in \mathcal{S}_\oplus$  such that  $sRt$ , for every  $E$  equivalence class of  $R$ , for all action  $l$ :  $\mathcal{P}_\oplus(s, l, E) = \mathcal{P}_\oplus(t, l, E)$ . Indeed, let  $s, t$  be such that  $sRt$ . There are two possible cases:
  - There are  $\sigma$  a type, and  $M, N \in \mathcal{T}^\sigma$ , such that  $s = (M, \sigma)$ ,  $t = (N, \sigma)$ , and  $\llbracket M \rrbracket = \llbracket N \rrbracket$ . Let  $l$  be an action:
    - either  $l = eval$ . Then for every  $r \in \mathcal{S}_\oplus$ ,  $\mathcal{P}_\oplus(s, eval, r) > 0 \Leftrightarrow (r = (\hat{Z}, \sigma) \text{ and } Z \in \mathcal{S}(\llbracket r \rrbracket))$ . Let  $E$  be an equivalence class of  $R$ . By construction of  $R$ , we can see that:

- Or  $\exists \tau$ , such that  $E = \{\tau\} \times E_\tau$ , and since the element of  $E_\tau$  are not distinguished values,  $\mathcal{P}_\oplus(s, eval, E) = 0 = \mathcal{P}_\oplus(t, eval, E)$ .
- Or  $\exists \tau \neq \sigma, V \in \mathcal{V}^\tau$  such that  $E = \{(\hat{V}, \tau)\}$ , and  $\mathcal{P}_\oplus(s, eval, E) = 0 = \mathcal{P}_\oplus(t, eval, E)$ .
- Or  $\exists V \in \mathcal{V}^\sigma$ , such that  $E = \{(\hat{V}, \sigma)\}$ , and  $\mathcal{P}_\oplus(s, eval, E) = \llbracket M \rrbracket(V) = \llbracket N \rrbracket(V) = \mathcal{P}_\oplus(t, eval, E)$ .
- Or  $l \neq eval$ , and for all equivalence class  $E$  of  $R$ :  $\mathcal{P}_\oplus(s, eval, E) = 0 = \mathcal{P}_\oplus(t, eval, E)$ .
- $\exists \sigma$ , and  $V \in \mathcal{V}^\sigma$  such that  $M = (\hat{V}, \sigma) = N$ , and we have: for every  $E$  equivalence class of  $R$ , for every action  $l$ ,  $\mathcal{P}_\oplus(s, l, E) = \mathcal{P}_\oplus(t, l, E)$ .

□

As a consequence of the previous lemma, if  $M, N \in \mathcal{T}^\sigma$  are such that  $\llbracket M \rrbracket = \llbracket N \rrbracket$ , then  $M \sim_\sigma N$ .

**Example 6** For all  $\sigma, M, N$  such that  $\emptyset \vdash M, N : \sigma$  and  $\llbracket N \rrbracket = \emptyset$ , we have that  $M \lesssim_\sigma N$  implies  $\llbracket M \rrbracket = \emptyset$ . For every terms  $M, N$  such that  $x : \tau \vdash M : \sigma$ , and  $\emptyset \vdash N : \tau$ , we have, as a consequence of Lemma 3, that  $(\lambda x.M)N \sim_\sigma M[N/x]$ .

We have just defined applicative (bi)simulation as a family  $(\mathcal{R}_\sigma)_\sigma$ , each  $\mathcal{R}_\sigma$  being a relation on closed terms of type  $\sigma$ . We can extend it to a *typed* relation, by the usual open extension:

**Definition 16** 1. If  $\Gamma = x_1 : \tau_1, \dots, x_n : \tau_n$  is a context, a  $\Gamma$ -closure makes each variable  $x_i$  to correspond to a value  $V_i \in \mathcal{V}^{\tau_i}$  (where  $1 \leq i \leq n$ ). The set of  $\Gamma$ -closures is  $CC_\Gamma$ . For every term  $\Gamma \vdash M : \sigma$  and for every  $\Gamma$ -closure  $\xi$ ,  $M\xi$  is the term in  $\mathcal{T}^\sigma$  obtained by substituting the variables in  $\Gamma$  with the corresponding values from  $\xi$ .

2. Let be  $\mathcal{R} = (\mathcal{R}_\sigma) \in \mathcal{R}$ . We define the open extension of  $(\mathcal{R}_\sigma)$  as the typed relation  $\mathcal{R}_\circ = (\mathcal{P}_\sigma^\Gamma)$  where  $\mathcal{P}_\sigma^\Gamma \subseteq \mathcal{T}_\sigma^\Gamma \times \mathcal{T}_\sigma^\Gamma$  is defined by stipulating that  $M\mathcal{P}_\sigma^\Gamma N$  iff for every  $\xi \in CC_\Gamma$ ,  $(M\xi) \mathcal{R}_\sigma (N\xi)$ .

The following proposition say that  $\sim_\circ$  is exactly the intersection of  $\lesssim_\circ$  and of the opposite of  $\lesssim_\circ$ .

**Proposition 8**  $\Gamma \vdash M \sim_\circ N : \sigma$  iff  $\Gamma \vdash M \lesssim_\circ N : \sigma$  and  $\Gamma \vdash N \lesssim_\circ M : \sigma$ .

**Proof.** It follows from Proposition 7. □

**Lemma 4**  $\lesssim_\circ$  is a transitive and reflexive typed relation, and  $\sim_\circ$  is a transitive, reflexive and symmetric typed relation.

**Proof.** • Since  $\lesssim$  is a preorder on  $\mathcal{S}_\oplus$ ,  $\lesssim_\sigma$  is a preorder on  $\mathcal{T}^\sigma$ , too. By definition of  $\lesssim_\circ$ , we have the thesis.

- Since  $\sim$  is an equivalence relation on  $\mathcal{S}_\oplus$ ,  $\sim_\sigma$  is an equivalence relation on  $\mathcal{T}^\sigma$ , too. By definition of  $\sim_\circ$ , we have the thesis.

□

**Definition 17 (Simulation Preorder and Bisimulation Equivalence)** The typed relation  $\lesssim_\circ$  is said to be the simulation preorder. The typed relation  $\sim_\circ$  is said to be bisimulation equivalence.

## 4.4 Bisimulation Equivalence is a Congruence

In this section, we want to show that  $\sim_\circ$  is actually a congruence, and that  $\lesssim_\circ$  is a precongruence. In view of Proposition 8, it is enough to show that the typed relation  $\lesssim_\circ$  is a precongruence, since  $\sim_\circ$  is the intersection of  $\lesssim_\circ$  and the opposite relation of  $\lesssim_\circ$ . The key step consists in showing that  $\lesssim_\circ$  is compatible. This will be carried out by the Howe's Method, which is a general method for establishing such congruence properties [15].

The main idea of Howe's method consists in defining an auxiliary relation  $\lesssim_\circ^H$ , such that it is easy to see that it is compatible, and then prove that  $\lesssim_\circ = (\lesssim_\circ^H)^+$ .

**Definition 18** Let  $\mathcal{R}$  be a typed relation. We define inductively the typed relation  $\mathcal{R}^H$  by the rules of Figure 4.

$\frac{\Gamma, x : \sigma \vdash x R M : \sigma}{\Gamma \vdash MR^H N : \mathbf{int}}$	$\frac{\Gamma \vdash \underline{n}RM : \mathbf{int}}{\Gamma \vdash LR^H P : \mathbf{int}}$	$\frac{\Gamma \vdash \underline{bv}RM : \mathbf{bool}}{\Gamma \vdash \lambda x. N R L : A \rightarrow B}$
$\frac{\Gamma, x : \sigma \vdash x R^H M : \sigma}{\Gamma \vdash M \text{ op } LR^H R : \gamma}$	$\frac{\Gamma \vdash \underline{n}R^H M : \mathbf{int}}{\Gamma \vdash \text{fix } x. N R L : A}$	$\frac{\Gamma \vdash \underline{bv}R^H M : \mathbf{bool}}{\Gamma \vdash \lambda x. M R^H L : A \rightarrow B}$
$\frac{\Gamma \vdash M R^H N : \mathbf{int}}{\Gamma \vdash M R^H L : A}$	$\frac{\Gamma \vdash N \text{ op } P R R : \gamma_{\text{op}}}{\Gamma \vdash \langle L, P \rangle R R : A \times B}$	$\frac{\Gamma \vdash \lambda x. M R^H L : A \rightarrow B}{\Gamma \vdash M R^H N : A \rightarrow B}$
$\frac{\Gamma \vdash \text{fix } x. M R^H L : A}{\Gamma \vdash M R^H L : A}$	$\frac{\Gamma \vdash \langle L, P \rangle R R : A \times B}{\Gamma \vdash \text{fst}(P') R N : A}$	$\frac{\Gamma \vdash LR^H P : A}{\Gamma \vdash LR^H P : A}$
$\frac{\Gamma \vdash N R^H P : B}{\Gamma \vdash M R^H L : A}$	$\frac{\Gamma \vdash \langle M, N \rangle R^H R : A \times B}{\Gamma \vdash M R^H N : A}$	$\frac{\Gamma \vdash \text{fst}(P) R^H N : A}{\Gamma \vdash LR^H P : A}$
$\frac{\Gamma \vdash \text{fst}(P) R^H N : A}{\Gamma \vdash M R^H N : A}$	$\frac{\Gamma \vdash M R^H N : A}{\Gamma \vdash M R^H P : \mathbf{bool}}$	$\frac{\Gamma \vdash N \oplus P R R : A}{\Gamma \vdash N R^H R : A}$
$\frac{\Gamma \vdash \text{snd}(P) R^H N : B}{\Gamma \vdash M R^H P : \mathbf{bool}}$	$\frac{\Gamma \vdash N R^H R : A}{\Gamma \vdash N R^H R : A}$	$\frac{\Gamma \vdash M \oplus L R^H R : A}{\Gamma \vdash \text{if } P \text{ then } R \text{ else } T R U : A}$
$\frac{\Gamma \vdash H R^H H' : A}{\Gamma \vdash H R^H H' : A}$	$\frac{\Gamma \vdash T R^H T' : [A]}{\Gamma \vdash T R^H T' : [A]}$	$\frac{\Gamma \vdash H' :: T' R N : [A]}{\Gamma \vdash H' :: T' R N : [A]}$
$\frac{\Gamma \vdash H :: T R^H N : [A]}{\Gamma \vdash M_1 R^H M'_1 : A}$	$\frac{\Gamma \vdash H :: T R^H N : [A]}{\Gamma \vdash L R^H L' : [B]}$	$\frac{\Gamma \vdash H :: T R^H N : [A]}{\Gamma \vdash h :: B, t :: [B] \vdash M_2 R^H M'_2 : A}$
$\Gamma \vdash \text{case } L \text{ of } \{\text{nil} \rightarrow M_1 \mid h :: t \rightarrow M_2\} R^H N : A$		

Figure 4: Howe's Construction

We are now going to show, that if the relation  $\mathcal{R}$  we start from satisfies minimal requirements, namely that it is reflexive and transitive, then the transitive closure  $(\mathcal{R}^H)^+$  of the Howe's lifting is guaranteed to be a precongruence which contains  $\mathcal{R}$ . This is a direct consequence of the following results, whose proofs are standard inductions:

- Let  $\mathcal{R}$  be a reflexive typed relation. Then  $\mathcal{R}^H$  is a compatible.
- Let  $\mathcal{R}$  be transitive. Then :

$$(\Gamma \vdash M \mathcal{R}^H N : \sigma) \wedge (\Gamma \vdash N \mathcal{R} L : \sigma) \Rightarrow (\Gamma \vdash M \mathcal{R}^H L : \sigma) \quad (1)$$

- If  $\mathcal{R}$  is reflexive and  $\Gamma \vdash M \mathcal{R} N : \sigma$ , then  $\Gamma \vdash M \mathcal{R}^H N : \sigma$ .
- If  $\mathcal{R}$  is compatible, then so is  $\mathcal{R}^+$ .

We can now apply the Howe's construction to  $\lesssim_{\circ}$ , since it is clearly reflexive and transitive. The points above then tell us that  $\lesssim_{\circ}^H$ , and  $(\lesssim_{\circ}^H)^+$  are both compatible. What we are left with, then, is proving that  $(\lesssim_{\circ}^H)^+$  is also a simulation. Let be  $V \in \mathcal{V}^{\sigma}$ . For  $W \in \mathcal{V}^{\sigma \rightarrow \tau}$ , we will note  $g^V(W) = N[V/x]$  if  $M = \lambda x. N$ , and  $g^V(W) = (N[\text{fix } x. N/x]) V$  if  $M = \text{fix } x. N$ .

**Lemma 5** For every  $M, N$ ,  $(\emptyset \vdash M \lesssim_{\circ} N : \sigma \rightarrow \tau)$  implies  $(\emptyset \vdash g^V(M) \lesssim_{\circ} g^V(N) : \tau)$ .

**Proof.** It follows from the fact that  $(\hat{M}, \sigma \rightarrow \tau) \sim_v (\hat{N}, \sigma \rightarrow \tau)$ . □

**Lemma 6**  $\lesssim_{\circ}^H$  is value-substitutive: for every typing context  $\Gamma$  and for every terms  $M, N$  and values  $V, W$  such that  $\Gamma, x : A \vdash M \lesssim_{\circ}^H N : \sigma$  and  $\Gamma \vdash V \lesssim_{\circ}^H W : \tau$ , it holds that  $\Gamma \vdash M[V/x] \lesssim_{\circ}^H N[W/x] : \sigma$

**Lemma 7** For all  $M, N$  terms of  $\text{PCFL}_{\oplus}$

- If  $\emptyset \vdash M \lesssim_{\circ} N : \sigma \rightarrow \tau$ , then for every  $X \subseteq \mathcal{V}^{\sigma \rightarrow \tau}$ , it holds that  $\llbracket M \rrbracket(X) \leq \llbracket N \rrbracket(\lesssim_{\circ}(X))$ .
- If  $(\emptyset \vdash M \lesssim_{\circ} N : \sigma \times \tau)$ , then for every  $X \subseteq \mathcal{V}^{\sigma \times \tau}$  we have:  $\llbracket M \rrbracket(X) \leq \llbracket N \rrbracket(Y)$ , when  $Y = \{\langle L, P \rangle \mid \exists \langle R, T \rangle \in X \wedge \emptyset \vdash L \lesssim_{\circ} R : \sigma \wedge \emptyset \vdash P \lesssim_{\circ}^H T : \tau\}$ .
- If  $(\emptyset \vdash M \lesssim_{\circ} N : [\sigma])$  then  $\llbracket M \rrbracket(\text{nil}) \leq \llbracket N \rrbracket(\text{nil})$  and for every  $X \subseteq \mathcal{V}^{[\sigma]}$ ,  $\llbracket M \rrbracket(X) \leq \llbracket N \rrbracket(Y)$  where  $Y$  is the set of those terms  $K :: L$  such that  $\exists H, T$  with  $H :: T \in X$  and  $\emptyset \vdash \lesssim_{\circ} K : \sigma$ , and  $\emptyset \vdash T \lesssim_{\circ} L : [\sigma]$ .
- $\emptyset \vdash M \lesssim_{\circ} N : \mathbf{int} \Rightarrow \forall k \in \mathbb{N}, \llbracket M \rrbracket(k) \leq \llbracket N \rrbracket(k)$ .
- $\emptyset \vdash M \lesssim_{\circ} N : \mathbf{bool} \Rightarrow \forall b \in \mathbb{B}, \llbracket M \rrbracket(b) \leq \llbracket N \rrbracket(b)$ .

**Proof.** It follows from the definition of  $\lesssim_{\circ}$ . □

We also need an auxiliary, technical, lemma about probability assignments:

**Definition 19**  $\mathbb{P} = (\{p_i\}_{1 \leq i \leq n}, \{r_I\}_{I \subseteq \{1, \dots, n\}})$  is said to be a probability assignment if for each  $I \subseteq \{1, \dots, n\}$ , it holds that  $\sum_{i \in I} p_i \leq \sum_{J \cap I \neq \emptyset} r_J$ .

**Lemma 8 (Disentangling Sets)** Let  $P = (\{p_i\}_{1 \leq i \leq n}, \{r_I\}_{I \subseteq \{1, \dots, n\}})$  be a probability assignment. Then for every non-empty  $I \subseteq \{1, \dots, n\}$ , and for every  $k \in I$ , there is  $s_{k,I} \in [0, 1]$  satisfying the following conditions:

- for every  $I$ , it holds that  $\sum_{k \in I} s_{k,I} \leq 1$ ;
- for every  $k \in 1, \dots, n$ , it holds that  $p_k \leq \sum_{k \in I} s_{k,I} \cdot r_I$ .

**Proof.** Any probability assignment  $\mathbb{P}$  can be seen as a flow network, where nodes are the nonempty subsets of  $\{1, \dots, n\}$ , plus a distinguished source  $s$  and a distinguished target  $t$ . Edges, then, go from  $s$  to each singleton  $\{i\}$  (with capacity  $p_i$ ), from every nonempty  $I$  to  $I \cup \{i\} \subseteq \{1, \dots, n\}$  whenever  $i \notin I$  (with capacity 1) and from every such  $I$  to  $t$  (with capacity  $r_I$ ). The thesis, then, is easily proved equivalent to showing that such a net supports a flow equals to  $\sum p_i$ . And, indeed, the fact that this is the maximum flow from  $s$  to  $t$  can be proved by way of the Max-Flow Min-Cut Theorem.  $\square$

**Lemma 9 (Key Lemma)** For every terms  $M, N$ ,

- If  $\emptyset \vdash M \lesssim_{\circ}^H N : \sigma \rightarrow \tau$ , then for every  $X_1 \subseteq \mathcal{T}_{x:\sigma}^{\tau}$  and  $X_2 \subseteq \mathcal{T}_{x:\sigma \rightarrow \tau}^{\tau}$ , it holds that  $\llbracket M \rrbracket (\lambda x. X_1 \cup \text{fix } x. X_2) \leq \llbracket N \rrbracket (\lesssim_{\circ} (\lambda x. Y_1 \cup \text{fix } x. Y_2))$ , where  $Y_1 = \{L \in \mathcal{T}_{x:\sigma}^{\tau} \mid \exists P \in X_1. x : \sigma \vdash P \lesssim_{\circ}^H L : \tau\}$  and  $Y_2 = \{L \in \mathcal{T}_{x:\sigma \rightarrow \tau}^{\tau} \mid \exists P \in X_2. x : \sigma \rightarrow \tau \vdash P \lesssim_{\circ}^H L : \sigma \rightarrow \tau\}$ .
- If  $\emptyset \vdash M \lesssim_{\circ}^H N : \sigma \times \tau$ , then for every  $X \subseteq \mathcal{V}^{\sigma \times \tau}$  we have:  $\llbracket M \rrbracket (X) \leq \llbracket N \rrbracket (\lesssim_{\circ} (Y))$ , where  $Y = \{\langle L, P \rangle \mid \exists \langle R, T \rangle \in X \wedge \emptyset \vdash R \lesssim_{\circ}^H L : \sigma \wedge \emptyset \vdash T \lesssim_{\circ}^H P : \tau\}$ .
- If  $(\emptyset \vdash M \lesssim_{\circ}^H N : [\sigma])$  then it holds that  $\llbracket M \rrbracket (\text{nil}) \leq \llbracket N \rrbracket (\text{nil})$  and for every  $X \subseteq \mathcal{V}^{[\sigma]}$ ,  $\llbracket M \rrbracket (X) \leq \llbracket N \rrbracket (\lesssim_{\circ} (Y))$  where  $Y$  is the set of those  $K :: L$  such that there are  $H, T$  with  $H :: T \in X$ ,  $\emptyset \vdash H \lesssim_{\circ}^H K : \sigma$ , and  $\emptyset \vdash T \lesssim_{\circ}^H L : [\sigma]$ .
- $\emptyset \vdash M \lesssim_{\circ}^H N : \text{int} \Rightarrow \forall k \in \mathbb{N}, \llbracket M \rrbracket (k) \leq \llbracket N \rrbracket (k)$ .
- $\emptyset \vdash M \lesssim_{\circ}^H N : \text{bool} \Rightarrow \forall b \in \mathbb{B}, \llbracket M \rrbracket (b) \leq \llbracket N \rrbracket (b)$ .

**Proof.** We are first going to show a similar result for the approximation semantics, and then we will extend it to the actual semantics : let  $M$  be a well-typed term, and  $\mathcal{D}$  a value distribution such that  $M \Downarrow \mathcal{D}$ . Then  $\mathcal{D}$  verifies:

- If  $\emptyset \vdash M \lesssim_{\circ}^H N : \sigma \rightarrow \tau$ , then for every  $X_1 \subseteq \mathcal{T}_{x:\sigma}^{\tau}$  and  $X_2 \subseteq \mathcal{T}_{x:\sigma \rightarrow \tau}^{\tau}$ , it holds that  $\mathcal{D} (\lambda x. X_1 \cup \text{fix } x. X_2) \leq \llbracket N \rrbracket (\lesssim_{\circ} (\lambda x. Y_1 \cup \text{fix } x. Y_2))$ , where  $Y_1 = \{L \in \mathcal{T}_{x:\sigma}^{\tau} \mid \exists P \in X_1. x : \sigma \vdash P \lesssim_{\circ}^H L : \tau\}$  and  $Y_2 = \{L \in \mathcal{T}_{x:\sigma \rightarrow \tau}^{\tau} \mid \exists P \in X_2. x : \sigma \rightarrow \tau \vdash P \lesssim_{\circ}^H L : \sigma \rightarrow \tau\}$ .
- If  $\emptyset \vdash M \lesssim_{\circ}^H N : \sigma \times \tau$  then for every  $X \subseteq \mathcal{V}^{\sigma \times \tau}$  it holds that  $\mathcal{D}(X) \leq \llbracket N \rrbracket (\lesssim_{\circ} (Y))$ , when  $Y = \{\langle L, P \rangle \mid \exists \langle R, T \rangle \in X \wedge \emptyset \vdash L \lesssim_{\circ}^H R : \sigma \wedge \emptyset \vdash P \lesssim_{\circ}^H T : \tau\}$ .
- If  $\emptyset \vdash M \lesssim_{\circ}^H N : [\sigma]$  then it holds that  $\mathcal{D}(\text{Nil}) \leq \llbracket N \rrbracket (\text{Nil})$  and that for every  $X \subseteq \mathcal{V}^{[\sigma]} \setminus \{\text{Nil}\}$ ,  $\mathcal{D}(X) \leq \llbracket N \rrbracket (\lesssim_{\circ} (E))$  when

$$E = \{K :: L \text{ such that } \exists H, T \text{ with } H :: T \in X \text{ and } (\emptyset \vdash H \lesssim_{\circ}^H K : \sigma) \text{ and } (\emptyset \vdash T \lesssim_{\circ}^H L : [\sigma])\}$$

- $\emptyset \vdash M \lesssim_{\circ}^H N : \text{int} \Rightarrow \forall k \in \mathbb{N}, \mathcal{D}(k) \leq \llbracket N \rrbracket (k)$ .
- $\emptyset \vdash M \lesssim_{\circ}^H N : \text{bool} \Rightarrow \forall b \in \mathbb{B}, \mathcal{D}(b) \leq \llbracket N \rrbracket (b)$ .

We are going to show this thesis by induction on the structure of the logic derivation of  $M \Downarrow \mathcal{D}$ . We consider separately every possible rule which can be applied at the bottom of the derivation :

- If the rule used correspond to the fact that, for every term, the empty distribution is an approximate semantics, the derivation is :  $\frac{}{M \Downarrow \emptyset} b_v$ . Then  $\mathcal{D} = \emptyset$ , and for all set of values  $X$ ,  $\mathcal{D}(X) = 0$ , and it concludes the proof.
- If  $M$  is a value  $V$ , and that the last rule of the derivation is :  $\frac{}{V \Downarrow \{V^1\}} b_v$ . Then  $\mathcal{D} = \{V^1\}$  is the dirac distribution for  $V$ . We are now going to consider the different possible cases for the type of  $M$ . Some interesting cases are:

- If  $M$  is of type **int**, it means in fact that there exists an integer  $k$  such that  $M = \underline{k}$ . Since the only possible way to show  $(\emptyset \vdash \underline{k} \lesssim_{\circ}^H N : \mathbf{int})$  is :  $\frac{\emptyset \vdash \underline{k} \lesssim_{\circ} N : \mathbf{int}}{\emptyset \vdash \underline{k} \lesssim_{\circ}^H N : \mathbf{int}}$ , we can see that  $\emptyset \vdash \underline{k} \lesssim_{\circ} N : \mathbf{int}$ . By Lemma 7, it implies that  $\llbracket N \rrbracket = \{\underline{k}^1\} = \mathcal{D}$ , and it leads to the result.
- If  $M$  is of type **bool**, it means that there exists a boolean  $b$  such that  $M = \underline{b}$ , and it's similar to the previous case.
- If  $M$  is of functional type  $\sigma \rightarrow \tau$ ,  $M$  can be either an abstraction  $\lambda x.L$ , or of the form  $\text{fix } x.L$ . We first consider the situation where  $M$  is an abstraction  $\lambda x.L$ . The only possible way to show  $(\emptyset \vdash \lambda x.L \lesssim_{\circ}^H N : \sigma \rightarrow \tau)$  is :

$$\frac{x : \sigma \vdash L \lesssim_{\circ}^H P : \tau \quad \emptyset \vdash \lambda x.P \lesssim_{\circ} N : \sigma \rightarrow \tau}{\emptyset \vdash \lambda x.L \lesssim_{\circ}^H N : \sigma \rightarrow \tau}$$

As  $(\emptyset \vdash \lambda x.P \lesssim_{\circ} N : \sigma \rightarrow \tau)$ , we can see by Lemma 7 :  $1 = \llbracket N \rrbracket(\lesssim_{\circ} \{\lambda x.P\})$ .

Let  $X_1 \subseteq \mathcal{T}_{x:\sigma}^{\tau}$  and  $X_2 \subseteq \mathcal{T}_{x:\sigma \rightarrow \tau}^{\sigma \rightarrow \tau}$ . We define  $Y_1$  as the set of terms which are greater or equal to a term in the set  $X_1$  by the Howe's lifting of the relation  $\lesssim_{\circ}$ , that is  $Y_1 = \{L \in \mathcal{T}_{x:\sigma}^{\tau} \mid \exists R \in X_1.x : \sigma \vdash R \lesssim_{\circ}^H L : \tau\}$  and we define analogously  $Y_2$  as the same thing for the set  $X_2$ , that is  $Y_2 = \{L \in \mathcal{T}_{x:\sigma \rightarrow \tau}^{\sigma \rightarrow \tau} \mid \exists R \in X_2.x : \sigma \rightarrow \tau \vdash R \lesssim_{\circ}^H L : \sigma \rightarrow \tau\}$ . We define  $Y = (\lambda x.Y_1 \cup \text{fix } x.Y_2)$ , and our aim is to show:  $\mathcal{D}(\lambda x.X_1 \cup \text{fix } x.X_2) \leq \llbracket N \rrbracket(\lesssim_{\circ}(Y))$ . Since  $\mathcal{D}$  is the dirac distribution for  $\lambda x.L$ , we have :

$$\mathcal{D}(\lambda x.X_1 \cup \text{fix } x.X_2) = \begin{cases} 0 & \text{if } L \notin X_1 \\ 1 & \text{otherwise} \end{cases}$$

If  $L \notin X_1$ , then  $\mathcal{D}(\lambda x.X_1 \cup \text{fix } x.X_2) = 0 \leq \llbracket N \rrbracket(\lesssim_{\circ}(Y))$ , and the thesis holds. If  $L \in X_1$ , then:  $\mathcal{D}(\lambda x.X_1 \cup \text{fix } x.X_2) = 1 = \llbracket N \rrbracket(\lesssim_{\circ}(\{\lambda x.P\}))$ . To conclude, we need to have:  $(\lesssim_{\circ}(\{\lambda x.P\}) \subseteq \lesssim_{\circ}(Y))$ . In fact, it is enough to show that  $\lambda x.P \in Y$ , and this is true since  $P \in Y_1$ .

The proof is similar if  $M = \text{fix } x.L$ .

- If  $M$ 's type is a product type  $\sigma_1 \times \sigma_2$ , it means that there exist  $L_1$  and  $L_2$ , such that  $M = \langle L_1, L_2 \rangle$ . Then the logical derivation of  $\emptyset \vdash \langle L_1, L_2 \rangle \lesssim_{\circ}^H N : \sigma_1 \times \sigma_2$  must be of the following form:

$$\frac{\emptyset \vdash L_1 \lesssim_{\circ}^H P_1 : \sigma_1 \quad \emptyset \vdash L_2 \lesssim_{\circ}^H P_2 : \sigma_2 \quad \emptyset \vdash \langle P_1, P_2 \rangle \lesssim_{\circ} N : \sigma_1 \times \sigma_2}{\emptyset \vdash \langle L_1, L_2 \rangle \lesssim_{\circ}^H N : \sigma_1 \times \sigma_2}$$

Now we can exploit the fact that  $\langle P_1, P_2 \rangle$  is lesser or equal than  $N$  for the relation  $\lesssim_{\circ}$  : we use the property of  $\lesssim_{\circ}$  stated in Lemma 7, and we obtain that :  $1 = \llbracket N \rrbracket(\{\langle U_1, U_2 \rangle \text{ s.t. } \emptyset \vdash P_i \lesssim_{\circ} U_i : \sigma_i \text{ for } i = 1, 2\})$ .

Moreover the Howe's lifting of a relation  $\mathcal{R}$  verifies, by (1) : if  $a\mathcal{R}^H b$  and  $b\mathcal{R}c$ , then  $a\mathcal{R}^H c$ . We can apply this property of Howe's lifting to see that : for every pair  $T = \langle R_1, R_2 \rangle$  such that  $\emptyset \vdash P_i \lesssim_{\circ} R_i$  for  $i = 1, 2$ , we have that  $L_i \lesssim_{\circ}^H R_i : \sigma_i$  for  $i = 1, 2$ . So, we have:  $T \in Z = \{\langle U_1, U_2 \rangle \text{ s.t. } \emptyset \vdash L_i \lesssim_{\circ}^H U_i : \sigma_i \text{ for } i = 1, 2\}$ . And so  $\{\langle U_1, U_2 \rangle \text{ s.t. } \emptyset \vdash P_i \lesssim_{\circ} U_i : \sigma_i \text{ for } i = 1, 2\} \subseteq Z$ , and consequently :

$$1 \leq \llbracket N \rrbracket(\{\langle U_1, U_2 \rangle \text{ s.t. } \emptyset \vdash P_i \lesssim_{\circ} U_i : \sigma_i \text{ for } i = 1, 2\}) \leq \llbracket N \rrbracket(Z).$$

Let be  $X$  a set of values of type  $\sigma_1 \times \sigma_2$ . Let be  $Y = \{\langle R_1, R_2 \rangle \mid \exists \langle T_1, T_2 \rangle \in X \wedge \emptyset \vdash R_i \lesssim_{\circ}^H T_i : \sigma_i \text{ for } i = 1, 2\}$ . We are aiming to show that :  $\mathcal{D}(X) \leq \llbracket N \rrbracket(\lesssim_{\circ}(Y))$ .

Since  $\mathcal{D}$  is the dirac distribution for  $\langle L_1, L_2 \rangle$ , if  $\langle L_1, L_2 \rangle$  is not in  $X$ , then  $\mathcal{D}(X) = 0$ , and it concludes the proof. Otherwise,  $\langle L_1, L_2 \rangle \in X$ , and then:

$$\mathcal{D}(X) = 1 = \llbracket N \rrbracket(Z) \leq \llbracket N \rrbracket(\{\langle U_1, U_2 \rangle \mid \exists \langle R_1, R_2 \rangle \in X \text{ s.t. } \emptyset \vdash R_i \lesssim_{\circ}^H U_i : \sigma_i \text{ for } i = 1, 2\})$$

, which is the thesis.

- If the derivation of  $M \Downarrow \mathcal{D}$  is of the following form:



$$\frac{M_1 \Downarrow \mathcal{K} \quad M_2 \Downarrow \mathcal{F} \quad \{P[V/x] \Downarrow \mathcal{E}_{P,V}\}_{\lambda x.P \in \mathcal{S}(\mathcal{K}), V \in \mathcal{S}(\mathcal{F})} \quad \{Q[\text{fix } x.Q/x]V \Downarrow \mathcal{G}_{Q,V}\}_{\text{fix } x.Q \in \mathcal{S}(\mathcal{K}), v \in \mathcal{S}(\mathcal{F})}}{M_1 M_2 \Downarrow \sum_{V \in \mathcal{S}(\mathcal{F})} \mathcal{F}(V) \left( \sum_{\lambda x.P \in \mathcal{S}(\mathcal{K})} \mathcal{K}(\lambda x.P) \cdot \mathcal{E}_{P,V} + \sum_{\text{fix } x.Q \in \mathcal{S}(\mathcal{K})} \mathcal{K}(\text{fix } x.Q) \cdot \mathcal{G}_{Q,V} \right)}$$

Then  $M = M_1 M_2$ . Let us suppose that:  $\emptyset \vdash M \lesssim_{\circ}^H N : \tau$ . The last rule used to prove this should be:

$$\frac{\emptyset \vdash M_1 \lesssim_{\circ}^H M'_1 : \sigma \rightarrow \tau \quad \emptyset \vdash M_2 \lesssim_{\circ}^H M'_2 : \sigma \quad \emptyset \vdash M'_1 M'_2 \lesssim_{\circ} N : \tau}{\Gamma \vdash M_1 M_2 \lesssim_{\circ}^H N : \tau}$$

We are first going to apply the induction hypothesis to the derivation of  $M_1 \Downarrow \mathcal{K}$ . The support of the value distribution  $\mathcal{K}$  is a finite set. We can note:  $\mathcal{S}(\mathcal{K}) = V_1, \dots, V_m$ . We can in fact separate it into two finite sets: a set  $A$  consisting in values in abstraction form, and a set  $B$  consisting in values in  $(\text{fix } x. \cdot)$  form. Let  $P_1, \dots, P_m$  defined by: if  $V_i$  is an abstraction form, then  $V_i = \lambda x.P_i$ , otherwise  $V_i = \text{fix } x.P_i$ . For each value in abstraction form  $\lambda x.P_i$ , we define the set  $(K_i = \{\lambda x.L \mid x : \sigma \vdash P_i \lesssim_{\circ}^H L : \tau\})$ , and on the same way, for each value  $\text{fix } x.Q_j$  we define the set  $(K_j = \{\text{fix } x.L \mid x : \sigma \rightarrow \tau \vdash Q_j \lesssim_{\circ}^H L : \sigma \rightarrow \tau\})$ . Now we can apply the induction hypothesis to the derivation of  $M_1 \Downarrow \mathcal{K}$ . Since we know that  $\emptyset \vdash M_1 \lesssim_{\circ}^H M'_1 : \sigma \rightarrow \tau$ , the induction hypothesis allows us to see that:

$\forall I \subseteq \{1, \dots, m\}$

$$\mathcal{K} \left( \bigcup_{i \in I} \{V_i\} \right) \leq \llbracket M'_1 \rrbracket \left( \bigcup_{i \in I} (\lesssim_{\circ} K_i) \right) \quad (2)$$

(2) allows us to apply Lemma 8: let's stipulate that  $n = m$ ,  $p_i = \mathcal{K}(V_i)$ . We define  $r_S = \sum_{W \text{ s.t. } \{j \mid W \in K_j\} = S} \llbracket M'_2 \rrbracket(W)$ . We can see that  $((p_i)_{1 \leq i \leq n}, (r_J)_{J \subseteq \{1, \dots, n\}})$  is a probabilistic assignment. So we can conclude, by applying Lemma 8 that:

for every  $U \in \bigcup_{1 \leq i \leq m} (\lesssim_{\circ} K_i)$ , there exist  $m$  real numbers  $r_1^U, \dots, r_m^U$ , such that:

$$\begin{aligned} \llbracket M'_1 \rrbracket(U) &\geq \sum_{1 \leq i \leq m} r_i^U & \forall U \in \bigcup_{1 \leq i \leq m} K_i \\ \mathcal{K}(V_i) &\leq \sum_{U \in K_i} r_i^U & \forall 1 \leq i \leq m \end{aligned}$$

In the same way, we can apply the induction hypothesis to  $M_2$ : Let be  $\mathcal{S}(\mathcal{F}) = \{V_1, \dots, V_l\}$ . Let be  $X_k = \lesssim_{\circ}^H (V_k)$ . We have by induction hypothesis:  $\forall I \subseteq \{1, \dots, l\}$ ,  $\mathcal{F}(\{V_k \mid k \in I\}) \leq \llbracket M'_2 \rrbracket(\bigcup_{k \in I} X_k)$ . So for all  $W \in \bigcup_{1 \leq k \leq l} X_k$ , there exist  $l$  real numbers  $s_1^W, \dots, s_l^W$ , such that:

$$\begin{aligned} \llbracket M'_2 \rrbracket(W) &\geq \sum_{1 \leq k \leq l} s_k^W & \forall W \in \bigcup_{1 \leq k \leq l} X_k \\ \mathcal{F}(V_k) &\leq \sum_{W \in X_k} s_k^W & \forall 1 \leq k \leq l \end{aligned}$$

For every value  $Z \in \mathcal{V}^{\tau}$ , we are considering  $\mathcal{D}(Z)$ , that is the probability (associated to  $\mathcal{D}$ , as an approximate semantics of  $M$ ) to obtain the value  $Z$ :

$$\begin{aligned} \mathcal{D}(Z) &= \sum_{1 \leq k \leq l} \mathcal{F}(V_k) \left( \sum_{i \in A} \mathcal{K}(\lambda x.P_i) \cdot \mathcal{E}_{P_i, V_k}(Z) + \sum_{i \in B} \mathcal{K}(\text{fix } x.Q_i) \cdot \mathcal{G}_{Q_i, V_k}(Z) \right) \\ &= \sum_{1 \leq k \leq l} \left( \sum_{W \in X_k} s_k^W \right) \left( \sum_{i \in A} \left( \sum_{U \in K_i} r_i^U \right) \cdot \mathcal{E}_{P_i, V_k}(Z) + \sum_{j \in B} \left( \sum_{U \in K_j} r_j^U \right) \cdot \mathcal{G}_{Q_j, V_k}(Z) \right) \\ &= \sum_{1 \leq k \leq l} \left( \sum_{W \in X_k} s_k^W \left( \sum_{i \in A} \left( \sum_{U \in K_i} r_i^U \cdot \mathcal{E}_{P_i, V_k}(Z) \right) + \sum_{i \in B} \left( \sum_{U \in K_i} r_i^U \cdot \mathcal{G}_{Q_i, V_k}(Z) \right) \right) \right) \end{aligned}$$

Let be  $U \in \bigcup_{1 \leq i \leq m} \lesssim_{\circ}(K_i)$ , and  $W \in \bigcup_{1 \leq k \leq l} X_k$ . Let be  $L_{U,W} = T[W/x]$  if  $U = \lambda x.T$ , and  $L_{U,W} = (T[\text{fix } x.T/x]W)$  if  $U = \text{fix } x.T$ . We can suppose that  $\tau = \theta \times \delta$  (other cases are similar). Let be  $X \subseteq \mathcal{V}^{\tau}$ . Let be  $E(X) = \{\langle M, N \rangle \mid \exists L, P, \langle L, P \rangle \in X \text{ and } (\emptyset \vdash L \lesssim_{\circ}^H M : \theta), (\emptyset \vdash N \lesssim_{\circ}^H P : \delta)\}$ .

We are going to show:

- (i) If  $U \in \lesssim_{\circ}(K_i)$ , and  $i \in A$ , and  $W \in X_k$ ,  $\mathcal{E}_{P_i, V_k}(X) \leq \llbracket L_{U,W} \rrbracket(E(X))$ .
- (ii) If  $U \in \lesssim_{\circ}(K_j)$ , and  $j \in B$ , and  $W \in X_k$ ,  $\mathcal{G}_{Q_j, V_k}(X) \leq \llbracket L_{U,W} \rrbracket(E(X))$

- (i) Let be  $U \in \lesssim_{\circ}(K_i)$ ,  $i \in A$  and  $W \in X_k$ . Then there exists  $S$  such that

$$\emptyset \vdash \lambda x.S \lesssim_{\circ} U : \sigma \rightarrow \tau \quad (3)$$

$$x : \sigma \vdash P_i \lesssim_{\circ}^H S : \tau \quad (4)$$

And besides, since  $W \in X_k$ :

$$\emptyset \vdash V_k \lesssim_{\circ}^H W : \sigma \quad (5)$$

By (3), and Lemma 5, we have:  $\emptyset \vdash S[W/x] \lesssim_{\circ} L_{U,W}$ . Moreover, by (4), (5) and Lemma 6, we have  $\emptyset \vdash P_i[V_k/x] \lesssim_{\circ}^H S[W/x] : \tau$ .

So by (1), it follows:  $\emptyset \vdash P_i[V_k/x] \lesssim_{\circ}^H L_{U,W} : \tau$ . And, by induction hypothesis applied to  $P_i[V_k/x]$ , it implies that:  $\mathcal{E}_{P_i, V_k}(X) \leq \llbracket L_{U,W} \rrbracket(E(X))$ .

- (ii) The proof is similar to (i).

$$\begin{aligned} \mathcal{D}(X) &\leq \sum_{1 \leq k \leq l} \left( \sum_{W \in X_k} s_k^W \left( \sum_{i \in A} \left( \sum_{U \in K_i} r_i^U \cdot \mathcal{E}_{P_i, V_k}(X) \right) + \sum_{i \in B} \left( \sum_{U \in K_i} r_i^U \cdot \mathcal{G}_{Q_i, V_k}(X) \right) \right) \right) \\ &\leq \sum_{1 \leq k \leq l} \left( \sum_{W \in X_k} s_k^W \left( \sum_{i \in A} \left( \sum_{U \in K_i} r_i^U \cdot \llbracket L_{U,W} \rrbracket(E(X)) \right) + \sum_{i \in B} \left( \sum_{U \in K_i} r_i^U \cdot \llbracket L_{U,W} \rrbracket(E(X)) \right) \right) \right) \\ &\leq \sum_{W \in \left( \bigcup_{1 \leq k \leq l} X_k \right)} \sum_{U \in \left( \bigcup_{1 \leq i \leq m} K_i \right)} \left( \sum_{k \text{ s.t. } W \in X_k} s_k^W \right) \cdot \left( \sum_{i \text{ s.t. } U \in K_i} r_i^U \right) \llbracket L_{U,W} \rrbracket(E(X)) \\ &\leq \sum_{W \in \left( \bigcup_{1 \leq k \leq l} X_k \right)} \sum_{U \in \left( \bigcup_{1 \leq i \leq m} K_i \right)} (\llbracket M'_2 \rrbracket(W)) \cdot (\llbracket M'_1 \rrbracket(U)) \llbracket L_{U,W} \rrbracket(E(X)) \\ &\leq \llbracket M'_1 M'_2 \rrbracket(E(X)) \end{aligned}$$

□

A consequence of the Key Lemma, then, is that  $(\lesssim_{\circ}^H)^+$  is an applicative bisimulation, thus included in the largest one, namely  $\lesssim_{\circ}$ . Since the latter is itself included in  $\lesssim_{\circ}^H$ , we obtain that  $\lesssim_{\circ} = (\lesssim_{\circ}^H)^+$ . But  $(\lesssim_{\circ}^H)^+$  is a precongruence, and we get the main result of this section:

**Theorem 2 (Soundness)** *The typed relation  $\lesssim_{\circ}$  is a precongruence relation included in  $\leq$ . Analogously,  $\sim_{\circ}$  is a congruence relation included in  $\equiv$ .*

## 4.5 Back to Our Examples

We now have all the necessary tools to prove that the example programs from Section 2 are indeed context equivalent. As an example, let us consider again the following terms:

$$\begin{aligned} EXP_{FST} &= \lambda x. \lambda y. ENC \ x \ GEN : \mathbf{bool} \rightarrow \mathbf{bool} \rightarrow \mathbf{bool}; \\ EXP_{SND} &= \lambda x. \lambda y. ENC \ y \ GEN : \mathbf{bool} \rightarrow \mathbf{bool} \rightarrow \mathbf{bool}. \end{aligned}$$

One can define the relations  $\mathcal{R}_{\mathbf{bool}}$ ,  $\mathcal{R}_{\mathbf{bool} \rightarrow \mathbf{bool}}$ ,  $\mathcal{R}_{\mathbf{bool} \rightarrow \mathbf{bool} \rightarrow \mathbf{bool}}$  by stipulating that  $\mathcal{R}_\sigma = X_\sigma \times X_\sigma \cup ID_\sigma$  where

$$\begin{aligned} X_{\mathbf{bool}} &= \{(ENC \ \underline{\mathbf{true}} \ GEN), (ENC \ \underline{\mathbf{false}} \ GEN)\}; \\ X_{\mathbf{bool} \rightarrow \mathbf{bool}} &= \{(\lambda y. ENC \ y \ GEN), (\lambda y. ENC \ \underline{\mathbf{true}} \ GEN), (\lambda y. ENC \ \underline{\mathbf{false}} \ GEN)\}; \\ X_{\mathbf{bool} \rightarrow \mathbf{bool} \rightarrow \mathbf{bool}} &= \{EXP_{FST}, EXP_{SND}\}; \end{aligned}$$

and for every type  $\sigma$ ,  $ID_\sigma$  is the identity on  $\mathcal{T}^\sigma$ . When  $\sigma$  is not one of the types above,  $\mathcal{R}_\sigma$  can be set to be just  $ID_\sigma$ . This way, the family  $(\mathcal{R}_\sigma)$  can be seen as a relation  $R$  on the state space of  $\mathcal{M}_\oplus$  (since any state in the form  $(\hat{V}, \sigma)$  can be treated as  $(V, \sigma)$ ). But  $R$  is easily seen to be a bisimulation. Indeed:

- All pairs of terms in  $\mathcal{R}_{\mathbf{bool}}$  have the same semantics, since  $\llbracket ENC \ \underline{\mathbf{true}} \ GEN \rrbracket$  and  $\llbracket ENC \ \underline{\mathbf{false}} \ GEN \rrbracket$  are both the uniform distribution on the set of boolean values.
- The elements of  $X_{\mathbf{bool} \rightarrow \mathbf{bool}}$  are values, and if we apply any two of them to a fixed boolean value, we end up with two terms  $\mathcal{R}_{\mathbf{bool}}$  puts in relation.
- Similarly for  $X_{\mathbf{bool} \rightarrow \mathbf{bool} \rightarrow \mathbf{bool}}$ : applying any two elements of it to a boolean value yields two elements which are put in relations by  $X_{\mathbf{bool} \rightarrow \mathbf{bool}}$ .

Being an applicative bisimulation,  $(\mathcal{R}_\sigma)_\sigma$  is included in  $\sim$ . And, by Theorem 2, we can conclude that  $EXP_{FST} \equiv EXP_{SND}$ . Analogously, one can verify that  $EXP \equiv RND$ .

## 5 Full Abstraction

Theorem 2 tells us that applicative bisimilarity is a sound way to prove that certain terms are context equivalent. Moreover, applicative bisimilarity is a congruence, and can then be applied in any context yielding bisimilar terms. In this section, we ask ourselves *how close* bisimilarity and context equivalence really are. Is it that the two coincide?

### 5.1 LMPs, Bisimulation, and Testing

The concept of probabilistic bisimulation has been generalized to the continuous case by Edalat, Desharnais and Panangaden, more than ten years ago [8]. Similarity and bisimilarity as defined in the aforementioned paper were later shown to exactly correspond to appropriate, and relatively simple, notions of *testing* [30]. We will make essential use of this characterization when proving that context equivalence is included in bisimulation. And this section is devoted to giving a brief but necessary introduction to the relevant theory. For more details, please refer to [30] and to [4].

In the rest of this section,  $\mathcal{A}$  is a fixed set of labels. The first step consists in giving a generalization of LMCs in which the set of states is not restricted to be countable:

**Definition 20** *A labelled Markov process (LMP in the following) is a triple  $\mathcal{C} = (\mathcal{X}, \Sigma, \mu)$ , consisting of a set  $\mathcal{X}$  of states, a  $\sigma$ -field  $\Sigma$  on  $\mathcal{X}$ , and a transition probability function  $\mu : \mathcal{X} \times \mathcal{A} \times \Sigma \rightarrow [0, 1]$ , such that:*

- for all  $x \in \mathcal{X}$ , and  $a \in \mathcal{A}$ , the naturally defined function  $\mu_{x,a}(\cdot) : \Sigma \rightarrow [0, 1]$  is a subprobability measure;
- for all  $a \in \mathcal{A}$ , and  $A \in \Sigma$ , the naturally defined function  $\mu_{(\cdot),a}(A) : \mathcal{X} \rightarrow [0, 1]$  is measurable.

The notion of (bi)simulation can be smoothly generalized to the continuous case:

**Definition 21** *Let  $(\mathcal{X}, \Sigma, \mu)$  be a LMP, and let  $R$  be a reflexive relation on  $\mathcal{X}$ . We say that  $R$  is a simulation if it satisfies condition 1 below, and we say that  $R$  is a bisimulation if it satisfies both conditions 1 and 2:*

1. If  $x R y$ , then for every  $a \in \mathcal{A}$  and for every  $A \in \Sigma$  such that  $A = R(A)$ , it holds that  $\mu_{x,a}(A) \leq \mu_{y,a}(A)$ .
2. If  $x R y$ , then for every  $a \in \mathcal{A}$  and for every  $A \in \Sigma$ ,  $\mu_{x,a}(\mathcal{X}) = \mu_{y,a}(\mathcal{X})$ .

*We say that two states are bisimilar if they are related by some bisimulation.*

**Lemma 10** *Let  $\langle \mathcal{X}, \Sigma, \mu \rangle$  be a labelled Markov process.*

- There is a largest bisimulation on  $(\mathcal{X}, \Sigma, \mu)$  which is an equivalence relation.
- For an equivalence relation  $R$ , the two criteria in Definition 21 can be compressed into the following condition:  $xRy \Rightarrow (\forall a \in \mathcal{A})(\forall A \in \Sigma)(A = R(A) \Rightarrow \mu_{x,a}(A) = \mu_{y,a}(A))$ .

We will soon see that there is a natural way to turn any LMC into a LMP, in such a way that (bi)similarity stays the same. Before doing so, however, let us introduce the notion of a *test*:

**Definition 22** The test language  $\mathcal{T}$  is given by the grammar  $t ::= \omega \mid a \cdot t \mid \langle t, t \rangle$ , where  $a \in \mathcal{A}$ .

Please observe that tests are *finite* objects, and that there isn't any disjunctive nor any negative test in  $\mathcal{T}$ . Intuitively,  $\omega$  is the test which always succeeds, while  $\langle t, s \rangle$  corresponds to making two copies of the underlying state, testing them independently according to  $t$  and  $s$  and succeeding iff *both* tests succeed. The test  $a \cdot t$  consists in performing the action  $a$ , and in case of success perform the test  $t$ . This can be formalized as follows:

**Definition 23** Given a labelled Markov Process  $\mathcal{C} = (\mathcal{X}, \Sigma, \mu)$ , we define an indexed family  $\{P_{\mathcal{C}}(\cdot, t)\}_{t \in \mathcal{T}}$  (such that  $P_{\mathcal{C}}(\cdot, t) : \mathcal{X} \rightarrow \mathbb{R}$ ) by induction on the structure of  $t$ :

$$P_{\mathcal{C}}(x, \omega) = 1; \quad P_{\mathcal{C}}(x, a \cdot t) = \int P_{\mathcal{C}}(\cdot, t) d\mu_{x,a}; \quad P_{\mathcal{C}}(x, \langle t, s \rangle) = P_{\mathcal{C}}(x, t) \cdot P_{\mathcal{C}}(x, s).$$

From our point of view, the key result is the following one:

**Theorem 3 ([30])** Let  $\mathcal{C} = (\mathcal{X}, \Sigma, \mu)$  be a LMP. Then  $x, y \in \mathcal{X}$  are bisimilar iff  $P_{\mathcal{C}}(x, t) = P_{\mathcal{C}}(y, t)$  for every test  $t \in \mathcal{T}$ .

## 5.2 From LMPs to LMCs

We are now going to adapt Theorem 3 to LMCs, thus getting an analogous characterization of probabilistic bisimilarity for them.

Let  $\mathcal{M} = (\mathcal{X}, \mathcal{A}, \mathcal{P})$  be a LMC. The function  $\mu_{\mathcal{M}} : \mathcal{X} \times \mathcal{A} \times \mathcal{P}(\mathcal{X}) \rightarrow [0, 1]$  is defined by  $\mu_{\mathcal{M}}(s, a, X) = \sum_{x \in X} \mathcal{P}(s, a, x)$ . This construction allows us to see any LMC as a LMP:

**Lemma 11** Let  $\mathcal{M} = (\mathcal{X}, \mathcal{A}, \mathcal{P})$  be a LMC. Then  $(\mathcal{X}, \mathcal{P}(\mathcal{X}), \mu_{\mathcal{M}})$  is a LMP, that we denote as  $\mathcal{C}_{\mathcal{M}}$ .

**Proof.** •  $\mathcal{P}(\mathcal{X})$  is a  $\sigma$ -field (non-empty, closed under complementation and countable unions).

•  $\mu$  verifies :

– for every  $s \in \mathcal{X}$ , and  $a \in \mathcal{A}$ ,  $\mu_{s,a}$  is a sub-probability measure, since :

\*  $\mu_{s,a}(\emptyset) = \sum_{x \in \emptyset} \mathcal{P}(s, a, x) = 0$

\*  $\mu_{s,a}(\mathcal{X}) = \sum_{x \in \mathcal{X}} \mathcal{P}(s, a, x) \leq 1$  since  $\mathcal{P}$  is a probability matrix.

\* For all countable collection of pairwise disjoint  $A_n \in \mathcal{P}(\mathcal{X})$ ,

$$\begin{aligned} \mu_{s,a} \left( \bigcup_n A_n \right) &= \sum_{x \in (\bigcup_n A_n)} \mathcal{P}(s, a, x) \\ &= \sum_n \left( \sum_{x \in A_n} \mathcal{P}(s, a, x) \right) \\ &= \sum_n \mu_{s,a}(A_n) \end{aligned}$$

– for every  $a \in \mathcal{A}$ , and  $A \in \mathcal{P}(\mathcal{X})$ ,  $\mu_{-,a}(A) : \mathcal{X} \rightarrow [0, 1]$  is measurable since :  $\forall I \subseteq [0, 1]$ ,  $\mu_{-,a}(A)^{-1} \in \mathcal{P}(\mathcal{X})$ .

□

But how about bisimulation? Do we get the same notion of equivalence this way? The answer is positive:

**Lemma 12** *Let  $\mathcal{M} = (\mathcal{X}, \mathcal{A}, \mathcal{P})$  be a LMC, and let  $R$  be an equivalence relation over  $\mathcal{X}$ . Then  $R$  is a bisimulation with respect to  $\mathcal{M}$  if and only if  $R$  is a bisimulation with respect to  $\mathcal{C}_{\mathcal{M}}$ . Moreover, two states are bisimilar with respect to  $\mathcal{M}$  iff they are bisimilar with respect to  $\mathcal{C}_{\mathcal{M}}$ .*

**Proof.** Let  $\mathcal{R}$  be an equivalence relation over  $\mathcal{X}$ .

$\Rightarrow$  We suppose that  $\mathcal{R}$  is a bisimulation with respect to the Markov chain  $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ . By Lemma 10, it is enough to show that :

$$x\mathcal{R}y \Rightarrow (\forall a \in \mathcal{A})(\forall A \in \mathcal{P}(\mathcal{X}))(A = \mathcal{R}(A) \Rightarrow \mu_{x,a}(A) = \mu_{y,a}(A)).$$

Let  $x, y$  be such that  $x\mathcal{R}y$ . For every  $a \in \mathcal{A}$  and  $A \in \mathcal{P}(\mathcal{X})$  such that  $A = \mathcal{R}(A)$ , we have :  $A$  is a  $\mathcal{R}$ -equivalence class. So (since  $\mathcal{R}$  is a bisimulation with respect to the Markov chain),  $\mu_{x,a}(A) = \sum_{s \in A} \mathcal{P}(x, a, s) = \sum_{s \in A} \mathcal{P}(y, a, s) = \mu_{y,a}(A)$ .

$\Leftarrow$  We suppose that  $\mathcal{R}$  is a bisimulation with respect to the Markov process  $(\mathcal{X}, \mathcal{P}(\mathcal{X}), \mu)$ . Then  $\mathcal{R}$  is a bisimulation with respect to the Markov chain  $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ , since

- $\mathcal{R}$  is an equivalence relation.
- Let  $x, y$  be such that  $x\mathcal{R}y$ . For every  $a \in \mathcal{A}$ , and  $E$  a  $\mathcal{R}$ -equivalence class,

$$\begin{aligned} \sum_{s \in E} \mathcal{P}(x, a, s) &= \mu_{x,a}(E) \\ &= \mu_{y,a}(E) && \text{by Lemma 10} \\ &= \sum_{s \in E} \mathcal{P}(y, a, s) \end{aligned}$$

About the second statement:

$\Rightarrow$  Let  $x, y$  be two states which are bisimilar with respect to the Markov chain  $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ . Then there is a  $\mathcal{R}$  a bisimulation with respect to the Markov chain  $(\mathcal{X}, \mathcal{A}, \mathcal{P})$  such that  $x\mathcal{R}y$ . It follows from Lemma 12 that  $\mathcal{R}$  is a bisimulation with respect to the Markov Process  $(\mathcal{X}, \mathcal{P}(\mathcal{X}), \mu)$ . So,  $x$  and  $y$  are bisimilar with respect to the Markov process  $(\mathcal{X}, \mathcal{P}(\mathcal{X}), \mu)$ .

$\Leftarrow$  Let  $x, y$  be two states which are bisimilar with respect to the Markov Process  $(\mathcal{X}, \mathcal{P}(\mathcal{X}), \mu)$ . Then (by Lemma 10) we can consider  $\mathcal{R}$  the largest bisimulation (with respect to the Markov Process), and we know that  $\mathcal{R}$  is an equivalence relation. We have :  $x\mathcal{R}y$ . It follows from Lemma 12 that  $\mathcal{R}$  is a bisimulation with respect to the Markov chain, and so  $x$  and  $y$  are bisimilar with respect to the Markov Chain. □

Let  $\mathcal{M} = (\mathcal{X}, \mathcal{A}, \mathcal{P})$  be a LMC. We define an indexed family  $\{P_{\mathcal{M}}(\cdot, t)\}_{t \in \mathcal{T}}$  by  $P_{\mathcal{M}}(x, t) = P_{\mathcal{C}_{\mathcal{M}}}(x, t)$ , the latter being the function from Definition 23 applied to the Markov process  $\mathcal{C}_{\mathcal{M}}$ . As a consequence of the previous results in this section, we get that:

**Theorem 4** *Let  $\mathcal{M} = (\mathcal{X}, \mathcal{A}, \mathcal{P})$  be a LMC. Then two states  $x, y \in \mathcal{X}$  are bisimilar if and only if for all tests  $t \in \mathcal{T}$ ,  $P_{\mathcal{M}}(x, t) = P_{\mathcal{M}}(y, t)$ .*

**Proof.**

$x$  and  $y$  are bisimilar  $\Leftrightarrow x$  and  $y$  are bisimilar with respect to the Markov Process  $\mathcal{C}_{\mathcal{M}} = (\mathcal{X}, \mathcal{P}(\mathcal{X}), \mu)$

$$\Leftrightarrow \forall t \in \mathcal{T}, P_{\mathcal{C}_{\mathcal{M}}}(x, t) = P_{\mathcal{C}_{\mathcal{M}}}(y, t) \text{ by Theorem 3}$$

$$\Leftrightarrow \forall t \in \mathcal{T}, P_{\mathcal{M}}(x, t) = P_{\mathcal{M}}(y, t)$$

□

The last result derives appropriate expressions for the  $P_{\mathcal{M}}(\cdot, \cdot)$ , which will be extremely useful in the next section:

**Proposition 9** *Let  $\mathcal{M} = (\mathcal{X}, \mathcal{A}, \mathcal{P})$  be a LMC. For all  $x \in \mathcal{X}$ , and  $t \in \mathcal{T}$ , we have:*

$$P_{\mathcal{M}}(x, \omega) = 1; \quad P_{\mathcal{M}}(x, a \cdot t) = \sum_{s \in \mathcal{X}} \mathcal{P}(x, a, s) \cdot P_{\mathcal{M}}(s, t); \quad P_{\mathcal{M}}(x, \langle t, s \rangle) = P_{\mathcal{M}}(x, t) \cdot P_{\mathcal{M}}(x, s).$$

### 5.3 Every Test has an Equivalent Context

We are going to consider the labelled Markov Chain  $\mathcal{M}_{\oplus}$  defined previously. We know that two programs  $M$  and  $N$  in  $\mathcal{T}^{\sigma}$  are bisimilar if and only if the states  $(M, \sigma)$  and  $(N, \sigma)$  have exactly the same probability to succeed for the tests in  $\mathcal{T}$ , measured according to  $P_{\mathcal{M}}(\cdot, \cdot)$ . Proving that context equivalence is included in bisimulation boils down to show that if  $M$  and  $N$  have exactly the same convergence probability for all contexts, then they have exactly the same success probability for all tests. Or, more precisely, that for a given test  $t$ , and a given type  $\sigma$ , there exists a context  $C$ , such that for all term  $M$  of type  $\sigma$ , the success probability of  $t$  on  $(M, \sigma)$  is *exactly* the convergence probability of  $C[M]$ :

$$P_{\mathcal{M}_{\oplus}}((M, \sigma), t) = \sum \llbracket C[M] \rrbracket.$$

However, we should take into account states in the form  $(\hat{V}, \sigma) \in \mathcal{S}_{\oplus}$ , where  $V$  is a value. The formalisation of the just described idea is the following Lemma:

**Lemma 13** *Let  $\sigma$  be a type, and  $t$  a test. Then there are contexts  $C_t^{\sigma}$ , and  $D_t^{\sigma}$  such that  $\emptyset \vdash C_t^{\sigma}(\emptyset; \sigma) : \mathbf{bool}$ ,  $\emptyset \vdash D_t^{\sigma}(\emptyset; \sigma) : \mathbf{bool}$ , and for every  $M \in \mathcal{T}^{\sigma}$  and every  $V \in \mathcal{V}^{\sigma}$ , it holds that*

$$P_{\mathcal{M}_{\oplus}}((M, \sigma), t) = \sum \llbracket C_t^{\sigma}[M] \rrbracket; \quad P_{\mathcal{M}_{\oplus}}((\hat{V}, \sigma), t) = \sum \llbracket D_t^{\sigma}[V] \rrbracket.$$

**Proof.** We are going to show the thesis by induction on  $t$ .

- if  $t = \omega$ , then  $\forall \sigma$ , we define  $C_{\omega}^{\sigma} = (\lambda x. \mathbf{true})(\lambda z. [\cdot])$ , and  $D_{\omega}^{\sigma} = (\lambda x. \mathbf{true})(\lambda z. [\cdot])$ . And we have:

$$\forall \sigma, \forall M \in \mathcal{T}^{\sigma}, P_{\mathcal{M}_{\oplus}}((M, \sigma), \omega) = 1 = \sum (\llbracket C_{\omega}^{\sigma}[M] \rrbracket)$$

and

$$\forall \sigma, \forall V \in \mathcal{V}^{\sigma}, P_{\mathcal{M}_{\oplus}}((\hat{V}, \sigma), \omega) = 1 = \sum (\llbracket D_{\omega}^{\sigma}[V] \rrbracket).$$

- If  $t = \langle s_1, \dots, s_n \rangle$ . Let  $\sigma$  be a type. By induction hypothesis, for all  $1 \leq i \leq n$ , there exist  $C_{s_i}^{\sigma}$  and  $D_{s_i}^{\sigma}$ , such that

$$\emptyset \vdash C_{s_i}^{\sigma}(\sigma; \emptyset) : \mathbf{bool} \text{ and } \emptyset \vdash D_{s_i}^{\sigma}(\sigma; \emptyset) : \mathbf{bool}$$

$$\forall M \in \mathcal{T}^{\sigma}, P_{\mathcal{M}_{\oplus}}((M, \sigma), s_i) = \sum (\llbracket C_{s_i}^{\sigma}[M] \rrbracket)$$

$$\forall V \in \mathcal{V}^{\sigma}, P_{\mathcal{M}_{\oplus}}((\hat{V}, \sigma), s_i) = \sum (\llbracket D_{s_i}^{\sigma}[V] \rrbracket)$$

We define:

$$C_t^{\sigma} = (\lambda x. T)(\lambda z. [\cdot])$$

$$D_t^{\sigma} = (\lambda x. T^*)(\lambda z. [\cdot])$$

where:

$$T = \text{if } ((\lambda y. \mathbf{true})(C_{s_1}^{\sigma}[x \ 0]))$$

$$\text{then } ((\lambda y. \mathbf{true})(C_{s_2}^{\sigma}[x \ 0])) \text{ then } \dots \text{ (if } (\lambda y. \mathbf{true})(C_{s_n}^{\sigma}[x \ 0]) \text{ then } \mathbf{true} \text{ else } \mathbf{true}) \dots \text{ else } \mathbf{true}$$

else  $\mathbf{true}$

$$T^* = \text{if } ((\lambda y. \mathbf{true})(D_{s_1}^{\sigma}[x \ 0])) \text{ then } ((\lambda y. \mathbf{true})(D_{s_2}^{\sigma}[x \ 0])) \dots \text{ (if } (\lambda y. \mathbf{true})(D_{s_n}^{\sigma}[x \ 0]) \text{ then } \mathbf{true} \text{ else } \mathbf{true}) \dots \text{ else } \mathbf{true}$$

Let be  $M \in \mathcal{T}^\sigma$ . We have:

$$\begin{aligned}
\sum (\llbracket C_t^\sigma[M] \rrbracket) &= \prod_{1 \leq i \leq n} \sum (\llbracket C_{s_i}^\sigma[(\lambda z.M)\underline{0}] \rrbracket) \\
&= \prod_{1 \leq i \leq n} \sum (\llbracket C_{s_i}^\sigma[M] \rrbracket) \text{ since } \forall M \text{ closed term, } M \sim_{ctx} (\lambda z.M)\underline{0} \\
&= P_{\mathcal{M}_\oplus}((M, \sigma), s_1) \cdot \dots \cdot P_{\mathcal{M}_\oplus}((M, \sigma), s_n) \\
&= P_{\mathcal{M}_\oplus}((M, \sigma), \langle s_1, \dots, s_n \rangle)
\end{aligned}$$

And similarly we have, for every  $V \in \mathcal{V}^\sigma$ :

$$\sum (\llbracket D_t^\sigma[V] \rrbracket) = P_{\mathcal{M}_\oplus}((\hat{V}, \sigma), \langle s_1, \dots, s_n \rangle)$$

- if  $t = a \cdot s$ 
  - if  $a = eval$ , we define:

$$D_t^\sigma = (\lambda x.[\cdot])\Omega$$

and

$$C_t^\sigma = (\lambda x.(D_s^\sigma[x]))([\cdot])$$

And we have  $\forall M \in \mathcal{T}^\sigma$ :

$$\begin{aligned}
\sum (\llbracket C_t^\sigma[M] \rrbracket) &= \sum_{V \in \mathcal{V}^\sigma} \llbracket M \rrbracket(V) \cdot \sum (\llbracket D_s^\sigma[V] \rrbracket) \\
&= \sum_{V \in \mathcal{V}^\sigma} \llbracket M \rrbracket(V) \cdot P_{(\hat{V}, \sigma)}(s, \\
&\quad) = \sum_{e \in \mathcal{S}_\oplus} \mathcal{P}_\oplus((M, \sigma), eval, y) \cdot P_{\mathcal{M}_\oplus}(e, s) \\
&= P_{\mathcal{M}_\oplus}((M, \sigma), s)
\end{aligned}$$

- if  $a = V$ , with  $V \in \mathcal{V}$ . we define  $C_t^\sigma = (\lambda x.[\cdot])\Omega$ .
  - if  $\sigma = \tau_1 \rightarrow \tau_2$ , and  $V \in \mathcal{V}^{\tau_1}$ , then we define:

$$D_t^{\tau_1 \rightarrow \tau_2} = C_s^{\tau_2}[[\cdot]V]$$

- otherwise, we define:  $D_t^\sigma = (\lambda x.[\cdot])\Omega$ .
- if  $a = fst$ : we define  $C_t^\sigma = (\lambda x.[\cdot])\Omega$ .
  - if  $\sigma = \tau_1 \times \tau_2$  then we define:

$$D_t^{\tau_1 \times \tau_2} = C_s^{\tau_1}[\mathbf{fst}([\cdot])]$$

- otherwise, we define:  $D_t^\sigma = (\lambda x.[\cdot])\Omega$ .
- if  $a = snd$ : similar to the previous case.
- if  $a = hd$ : we define  $C_t^\sigma = (\lambda x.[\cdot])\Omega$ .
  - if  $\sigma = [\tau]$  then we define:

$$D_t^{[\tau]} = C_s^\tau[\mathbf{case}[\cdot] \text{ of } \{\mathbf{nil} \rightarrow \Omega \mid h :: t \rightarrow h\}]$$

- otherwise, we define:  $D_t^\sigma = (\lambda x.[\cdot])\Omega$ .
- if  $a = tl$ : we define  $C_t^\sigma = (\lambda x.[\cdot])\Omega$ .
  - if  $\sigma = [\tau]$  then we define:

$$D_t^{[\tau]} = C_s^{[\tau]}[\mathbf{case}[\cdot] \text{ of } \{\mathbf{nil} \rightarrow \Omega \mid h :: t \rightarrow t\}]$$

- otherwise, we define:  $D_t^\sigma = (\lambda x.[\cdot])\Omega$ .

- if  $a = \text{nil}$ : we define  $C_t^\sigma = (\lambda x. [\cdot])\Omega$ .
  - if  $\sigma = [\tau]$  then we define:

$$D_t^{[\tau]} = C_s^\gamma[\text{case } [\cdot] \text{ of } \{\text{nil} \rightarrow C_s^\gamma[\tilde{div}] \mid h :: t \rightarrow \Omega\}]$$

- otherwise, we define:  $D_t^\sigma = (\lambda x. [\cdot])\Omega$ .
- if  $a = k$ , with  $k \in \mathbb{N}$ .
  - if  $\sigma = \mathbf{int}$  we define:  $D_t^{\mathbf{int}} = \text{if } ([\cdot] = \underline{k}) \text{ then } \underline{\text{true}} \text{ else } \Omega$ .
  - otherwise:  $D_t^\sigma = (\lambda x. [\cdot])\Omega$

□

It follows from Lemma 13 that if two well-typed closed terms are context equivalent, they are bisimilar:

**Theorem 5** *Let  $M, N$  be terms such that  $\emptyset \vdash M \equiv N : \sigma$ . Then  $\emptyset \vdash M \sim_\circ N : \sigma$ .*

**Proof.** Let  $t$  be a test. We have that, since  $M \equiv N$ ,

$$P_{\mathcal{M}_\oplus}((M, \sigma), t) = \sum \llbracket C_t^\sigma[M] \rrbracket = \sum \llbracket C_t^\sigma[N] \rrbracket = P_{\mathcal{M}_\oplus}((N, \sigma), t),$$

where  $C_t^\sigma$  is the context from Lemma 13. By Theorem 4,  $(M, \sigma)$  and  $(N, \sigma)$  are bisimilar. So  $\emptyset \vdash M \sim_\circ N : \sigma$  which is the thesis. □

We can now easily extend this result to terms in  $\mathcal{T}_\sigma^\Gamma$ , which gives us Full Abstraction: bisimilarity and context equivalence indeed coincide.

**Theorem 6 (Full Abstraction)** *Let  $M$  and  $N$  be terms in  $\mathcal{T}_\sigma^\Gamma$ . Then  $\Gamma \vdash M \equiv N : \sigma$  iff  $\Gamma \vdash M \sim_\circ N : \sigma$ .*

**Proof.** There is only one inclusion to show (we know already that bisimilarity is included in context equivalence). We know that  $\equiv$  is value substitutive. We note  $\Gamma = x_1 : \tau_1, \dots, x_n : \tau_n$ . So for all  $V_1 \in \mathcal{V}^{\tau_1}, \dots, V_n \in \mathcal{V}^{\tau_n}$ , we have:  $\emptyset \vdash M[\bar{V}/\bar{x}] \equiv N[\bar{V}/\bar{x}] : \sigma$ . By Theorem 5, we have:  $\emptyset \vdash M[\bar{V}/\bar{x}] \sim_v N[\bar{V}/\bar{x}] : \sigma$ . And so by definition of the open extension:  $\Gamma \vdash M \sim_\circ N : \sigma$ . □

## 5.4 The Asymmetric Case

Theorem 6 establishes a precise correspondence between bisimulation and context equivalence. This is definitely not the end of the story — surprisingly enough, indeed, *simulation* and the contextual *preorder* do not coincide, and this section gives a counterexample, namely a pair of terms which can be compared in the context preorder but which are not similar.

Let us fix the following terms:

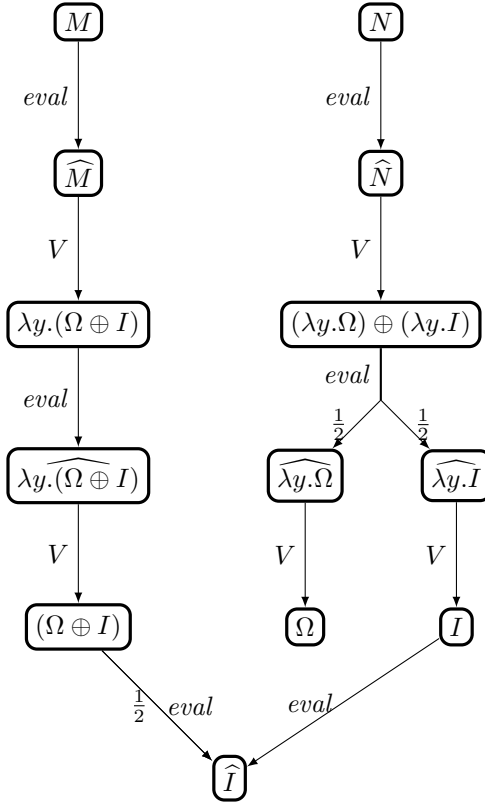
$$\begin{aligned} M &= \lambda x. \lambda y. (\Omega \oplus I); \\ N &= \lambda x. (\lambda y. \Omega) \oplus (\lambda y. I). \end{aligned}$$

Both these terms can be given the type  $\sigma = \mathbf{bool} \rightarrow \mathbf{bool} \rightarrow \mathbf{bool} \rightarrow \mathbf{bool}$  in the empty context. The first thing to note is that  $M$  and  $N$  cannot even be compared in the simulation preorder:

**Lemma 14** *It is not the case that  $\emptyset \vdash M \lesssim_\circ N : \sigma$  nor that  $\emptyset \vdash N \lesssim_\circ M : \sigma$ .*

**Proof.** The Markov Chain used to define  $\lesssim$  has the following form:





- Suppose that  $N \lesssim M$ . So (since  $\lesssim$  is a simulation),  $(\lambda x. I \oplus \lambda x. \Omega) \lesssim (\lambda x. (I \oplus \Omega))$ . So we have :  $\frac{1}{2} = \mathcal{P}_{\oplus}((\lambda x. I \oplus \lambda x. \Omega), eval, \widehat{\lambda x. I}) \leq \mathcal{P}_{\oplus}((\lambda x. (I \oplus \Omega)), eval, \lesssim (\lambda x. \widehat{I}))$ , and it folds that :  $(\lambda V. (I \oplus \Omega)) \in \lesssim (\lambda V. I)$ , i.e.  $(\lambda V. I \lesssim \lambda V. (I \oplus \Omega))$ . But since  $\lesssim$  is a simulation, we can then deduce that :  $I \lesssim I \oplus \Omega$  : and we have a contradiction since  $\mathcal{P}_{\oplus}(I, eval, \widehat{I}) > \mathcal{P}_{\oplus}(I \oplus \Omega, eval, \mathcal{S}_{\oplus})$ .
- Suppose that  $M \lesssim N$ . We use on the same way the fact that  $\lesssim$  is a simulation. We have :  $(\lambda x. (I \oplus \Omega)) \lesssim (\lambda x. I \oplus \lambda x. \Omega)$ . And so we have :  $1 = \mathcal{P}_{\oplus}((\lambda x. (I \oplus \Omega)), eval, \widehat{\lambda x. (I \oplus \Omega)}) \leq \mathcal{P}_{\oplus}((\lambda x. I \oplus \lambda x. \Omega), eval, \lesssim (\lambda x. \widehat{(I \oplus \Omega)}))$ . It implies that :  $\widehat{\lambda x. (I \oplus \Omega)} \lesssim \widehat{\lambda x. \Omega}$ . We can now apply the *eval* action, and we see that :  $\mathcal{P}_{\oplus}(I \oplus \Omega, eval, \mathcal{S}_{\oplus}) \leq \mathcal{P}_{\oplus}(I \oplus \Omega, eval, \mathcal{S}_{\oplus}) = 0$ , and so we have a contradiction.

This concludes the proof.  $\square$

We now proceed by proving that  $M$  and  $N$  can be compared in the *contextual* preorder. We will do so by studying their dynamics seen as terms of  $\Lambda_{\oplus}$  [6] (in which the only constructs are variables, abstractions, applications and probabilistic choices, and in which types are absent) rather than terms of  $\text{PCFL}_{\oplus}$ . We will later argue why this translates back into a result for  $\text{PCFL}_{\oplus}$ . This detour allows to simplify the overall treatment without sacrificing generality. From now on, then  $M$  and  $N$  are seen as pure terms, where  $\Omega$  takes the usual form  $(\lambda x. xx)(\lambda x. xx)$ .

Let us introduce some notation now. First of all, three terms need to be given names as follows:  $L = \lambda y. (\Omega \oplus I)$ ,  $L_0 = \lambda y. \Omega$ , and  $L_1 = \lambda y. I$ . If  $b = b_1, \dots, b_n \in \{0, 1\}^n$ , then  $L_b$  denotes the sequence of terms  $L_{b_1} \cdots L_{b_n}$ . If  $P$  is a term,  $P \Rightarrow^p$  means that there is distribution  $\mathcal{D}$  such that  $P \Rightarrow \mathcal{D}$  and  $\sum \mathcal{D} = p$  (where  $\Rightarrow$  is small-step approximation semantics [6]).

The idea, now, is to prove that in any term  $P$ , if we replace an occurrence of  $M$  by an occurrence of  $N$ , we obtain a term  $R$  which converges with probability smaller than the one with which  $P$  converges. We first need an auxiliary lemma, which proves a similar result for  $L_0$  and  $L_1$ .

**Lemma 15** *For every term  $P$ , if  $(P[L_0/x]) \Rightarrow^p$ , then there is another real number  $q \geq p$  such that  $(P[L_1/x]) \Rightarrow^q$ .*

**Proof.** First, we can remark that, for every term  $P$  and any variable  $z$  which doesn't appear in  $P$ ,  $P[L_0/x] = (P[\lambda y.z/x])[\Omega/z]$ , and  $P[L_1/x] = (P[\lambda y.z/x])[I/z]$ . It is thus enough to show that for every term  $R$ , if  $(R[\Omega/x]) \Rightarrow^p$ , then there is  $q \geq p$  such that  $(R[I/x]) \Rightarrow^q$ . This is an induction on the proof of  $(R[\Omega/x]) \Rightarrow^p$ , i.e., an induction on the structure of a derivation of  $(R[\Omega/x]) \Rightarrow \mathcal{D}$  where  $\sum \mathcal{D} = p$ . Some interesting cases:

- If  $(R[\Omega/x]) = V$  is a value, then the term  $(R[I/x])$  is a value too. So we have  $(R[I/x]) \Rightarrow \{(R[I/x])^1\}$ , and so  $(R[I/x]) \Rightarrow^1$ , and the thesis holds.
- Suppose that the derivation looks as follows:

$$\frac{(R[\Omega/x]) \rightarrow \bar{T} \quad T_i \Rightarrow \mathcal{E}_i}{(R[\Omega/x]) \Rightarrow \sum_{1 \leq i \leq k} \frac{1}{k} \cdot \mathcal{E}_i}$$

Then there are two possible cases :

- If  $R[\Omega/x] \rightarrow T_1, \dots, T_k$ , but the involved redex is *not*  $\Omega$ , then we can easily prove that each  $T_i$  can be written in the form  $U_i[\Omega/x]$ , where

$$R[\Omega/x] \rightarrow U_1[\Omega/x], \dots, U_k[\Omega/x].$$

Similarly  $R[I/x] \rightarrow U_1[I/x], \dots, U_k[I/x]$ . We can then apply the induction hypothesis to each of the derivations for  $U_i[\Omega/x]$ .

- The interesting case is when the active redex in  $R[\Omega/x]$  is  $\Omega$ . Since we have  $\Omega \rightarrow \Omega$ , we have  $R[\Omega/x] \rightarrow R[\Omega/x]$ , and so  $\bar{T} = T_1 = R[\Omega/x]$ , and  $\mathcal{D} = \mathcal{E}_1$ . We can apply the induction hypothesis to  $T_1 \Rightarrow \mathcal{E}_1$ , and the thesis follows.

This concludes the proof.  $\square$

We are now ready to prove the central lemma of this section, which takes a rather complicated form just for the sake of its inductive proof:

**Lemma 16** *Suppose that  $P$  is a term and suppose that  $(P[M, L/x, \bar{y}]) \Rightarrow^p$ , where  $\bar{y} = y_1, \dots, y_n$ . Then for every  $b \in \{0, 1\}^n$  there is  $p_b$  such that  $(P[N, L_b/x, \bar{y}]) \Rightarrow^{p_b}$  and  $\sum_b \frac{p_b}{2^n} \geq p$ .*

**Proof.** This is an induction on the proof of  $(P[M, L/x, \bar{y}]) \Rightarrow^p$ , i.e., an induction on the structure of a derivation of  $(P[M, L/x, \bar{y}]) \Rightarrow \mathcal{D}$  where  $\sum \mathcal{D} = p$ :

- If  $P[M, L/x, \bar{y}]$  is a value, then:
  - either  $p = 1$ , but we can also choose  $p_b$  to be 1 for every  $b$ , since the term  $P[N, L_b/x, \bar{y}]$  is a value, too;
  - or  $p = 0$ , and in this case we can fix  $p_b$  to be 0 for every  $b$ .
- If  $P[M, L/x, \bar{y}] \rightarrow R_1, \dots, R_k$ , but the involved redex has *not*  $M$  nor  $L$  as functions, then we are done, because one can easily prove in this case that each  $R_i$  can be written in the form  $T_i[M, L/x, \bar{y}]$ , where

$$P[N, L_b/x, \bar{y}] \rightarrow T_1[N, L_b/x, \bar{y}], \dots, T_k[N, L_b/x, \bar{y}].$$

It suffices, then, to apply the induction hypothesis to each of the derivations for  $T_i[M, L/x, \bar{y}]$ , easily reaching the thesis;

- The interesting case is when the active redex in  $P[M, L/x, \bar{y}]$  has either  $M$  or  $L$  (or, better, occurrences of them coming from the substitution) in functional position.
  - If  $M$  is involved, then there are a term  $R$  and a variable  $z$  such that

$$\begin{aligned} P[M, L/x, \bar{y}] &\rightarrow R[M, L, L/x, \bar{y}, z]; \\ P[N, L_b/x, \bar{y}] &\rightarrow R[N, L_b, L_0/x, \bar{y}, z], \rightarrow R[N, L_b, L_1/x, \bar{y}, z]. \end{aligned}$$

This, in particular, means that we can easily apply the induction hypothesis to  $R[M, L, L/x, \bar{y}, z]$ .

- If, on the other hand  $L$  is involved in the redex, then there are a term  $R$  and a variable  $z$  such that

$$P[M, L/x, \bar{y}] \rightarrow R[M, L, \Omega/x, \bar{y}, z], R[M, L, I/x, \bar{y}, z].$$

Moreover, the space of all sequences  $b$  can be partitioned into two classes of the same cardinality  $2^{n-1}$ , call them  $B_B$  and  $B_G$ ; for every  $b \in B_B$ , we have that  $P[N, L_b/x, \bar{y}]$  is diverging, while for every  $b \in B_G$ , we have that

$$P[N, L_b/x, \bar{y}] \rightarrow R[N, L_b, I/x, \bar{y}, z].$$

Observe how for any  $b \in B_B$  there is  $\hat{b} \in B_G$  such that  $b$  and  $\hat{b}$  agree on every bit except one, which is 0 in  $b$  and 1 in  $\hat{b}$ . Now, observe that  $p = \frac{q}{2}$  where  $R[M, L, I/x, \bar{y}, z] \Rightarrow^q$ . We can then apply the induction hypothesis and obtain that  $q \leq \sum_b \frac{q_b}{2^n}$  where  $R[N, L_b, I/x, \bar{y}, z] \Rightarrow^{q_b}$ . Due to Lemma 15, we can assume without losing generality that  $q_b \leq q_{\hat{b}}$  for every  $b \in B_B$ . Now, fix  $p_b = 0$  if  $b \in B_B$  and  $p_b = q_b$  if  $b \in B_G$ . Of course  $(P[N, L_b/x, \bar{y}]) \Rightarrow^{p_b}$ . But moreover,

$$p = \frac{q}{2} \leq \frac{1}{2} \sum_b \frac{q_b}{2^n} \leq \frac{1}{2} \sum_{b \in B_G} \frac{2 \cdot q_b}{2^n} = \sum_{b \in B_G} \frac{q_b}{2^n} = \sum_b \frac{p_b}{2^n}.$$

This concludes the proof.  $\square$

From what we have seen so far, it is already clear that for any context  $C$ , it cannot be that  $\sum[C[M]] > \sum[C[N]]$ , as this would mean that for a certain term  $P$ ,  $P[M/x]$  would converge to a distribution  $\mathcal{D}$  whose sum  $p$  is higher than the sum of any distribution to which  $P[N/x]$  converges, and this is in contradiction with Lemma 16: simply consider the case where  $n = 0$ .

But how about  $\text{PCFL}_\oplus$ ? Actually, there is an embedding  $\langle\langle \cdot \rangle\rangle$  of  $\text{PCFL}_\oplus$  into  $\Lambda_\oplus$  such that for every  $P \in \mathcal{T}^\sigma$ , it holds that  $\sum[P] = \sum[\langle\langle P \rangle\rangle]$  (for more details, see the next section). As a consequence there cannot be any  $\text{PCFL}_\oplus$  context contradicting what we have said in the last paragraph. Summing up,

**Theorem 7** *The simulation preorder  $\lesssim_\circ$  is not fully abstract.*

The careful reader may now wonder whether a result akin to Theorem 4 exists for *simulation* and testing. Actually, there *is* such a result [30], but for a different notion of test, which not only, like  $\mathcal{T}$ , includes conjunctive tests, but also disjunctive ones. Now, anybody familiar with the historical developments of the quest for a fully abstract model of PCF [26, 2] would immediately recognize disjunctive tests as something which cannot be easily implemented by terms.

## 5.5 Embedding $\text{PCFL}_\oplus$ into $\Lambda_\oplus$

The embedding  $\langle\langle \cdot \rangle\rangle$  maps any term in  $\text{PCFL}_\oplus$  into a pure, untyped, term. It is defined as follows:

$$\begin{aligned} \langle\langle x \rangle\rangle &= x; \\ \langle\langle n \rangle\rangle &= [n]; \\ \langle\langle b \rangle\rangle &= [b]; \\ \langle\langle \text{nil} \rangle\rangle &= \lambda x. \lambda y. x \star; \\ \langle\langle M :: N \rangle\rangle &= \lambda x. \lambda y. y \langle\langle M \rangle\rangle \langle\langle N \rangle\rangle; \\ \langle\langle M, N \rangle\rangle &= \lambda x. x (\lambda. \langle\langle M \rangle\rangle) (\lambda. \langle\langle N \rangle\rangle); \\ \langle\langle \lambda x. M \rangle\rangle &= \lambda x. \langle\langle M \rangle\rangle; \\ \langle\langle \text{fix } x. M \rangle\rangle &= \lambda y. M_{\text{fix}} (\lambda x. \langle\langle M \rangle\rangle) y; \\ \langle\langle M \oplus N \rangle\rangle &= \langle\langle M \rangle\rangle \oplus \langle\langle N \rangle\rangle; \\ \langle\langle \text{if } M \text{ then } N \text{ else } L \rangle\rangle &= \langle\langle M \rangle\rangle (\lambda. \langle\langle N \rangle\rangle) (\lambda. \langle\langle L \rangle\rangle) \star; \\ \langle\langle M \text{ op } N \rangle\rangle &= M_{\text{op}} \langle\langle M \rangle\rangle \langle\langle N \rangle\rangle; \\ \langle\langle \text{fst } (M) \rangle\rangle &= \langle\langle M \rangle\rangle (\lambda x. \lambda y. x) \star; \\ \langle\langle \text{snd } (M) \rangle\rangle &= \langle\langle M \rangle\rangle (\lambda x. \lambda y. y) \star; \\ \langle\langle MN \rangle\rangle &= \langle\langle M \rangle\rangle \langle\langle N \rangle\rangle; \\ \langle\langle \text{case } M \text{ of } \{ \text{nil} \rightarrow N \mid h :: t \rightarrow L \} \rangle\rangle &= \langle\langle M \rangle\rangle (\lambda. \langle\langle N \rangle\rangle) (\lambda h. \lambda t. L); \end{aligned}$$

where:

- $[\cdot]$  is the so-called Scott-encoding of natural numbers and booleans in the  $\lambda$ -calculus:
- $M_{\text{fix}}$  is the term  $NN$ , where  $N$  is the term  $\lambda x.\lambda y.y(\lambda z.((xx)y)z)$ .
- $M_{\text{op}}$  is the term implementing `op`, which we suppose to always exist given the universality of weak call-by-value reduction [5].

**Lemma 17** *For every  $\text{PCFL}_{\oplus}$  term  $M$ ,  $M$  is a value iff  $\langle\langle M \rangle\rangle$  is a value.*

**Lemma 18** *For every typable  $\text{PCFL}_{\oplus}$  term  $M$ , if  $M \Rightarrow \mathcal{D}$ , then  $\langle\langle M \rangle\rangle \Rightarrow \langle\langle \mathcal{D} \rangle\rangle$ .*

**Proposition 10** *For every typable  $\text{PCFL}_{\oplus}$  term  $M$ ,  $\langle\langle [M] \rangle\rangle = \langle\langle [M] \rangle\rangle$ .*

## 6 A Comparison with Call-by-Name

Actually,  $\text{PCFL}_{\oplus}$  could easily be endowed with call-by-name rather than call-by-value operational semantics. The obtained calculus, then, is amenable to a treatment similar to the one described in Section 4. Full abstraction, however, holds neither for simulation nor for bisimulation. These results are anyway among the major contributions of [10]. The precise correspondence between testing and bisimulation described in Section 5.2 shed some further light on the gap between call-by-value and call-by-name evaluation. In both cases, indeed, bisimulation can be characterized by testing as given in Definition 22. What call-by-name evaluation misses, however, is the capability to copy a term *after* having evaluated it, a feature which is instead available if parameters are passed to function evaluated, as in call-by-value. In a sense, then, the tests corresponding to bisimilarity are the same in call-by-name, but the calculus turns out to be too poor to implement all of them. We conjecture that the subclass of tests which are implementable in a call-by-name setting are those in the form  $\langle t_1, \dots, t_n \rangle$  (where each  $t_i$  is in the form  $a_i^1 \cdot \dots \cdot a_i^{m_i} \cdot \omega$ ), and that full abstraction can be recovered if the language is endowed with an operator for *sequencing*.

## 7 Conclusions

In this paper, we study probabilistic applicative bisimulation in a call-by-value scenario, in the meantime generalizing it to a typed language akin to Plotkin’s PCF. Actually, some of the obtained results turn out to be surprising, highlighting a gap between the symmetric and asymmetric cases, and between call-by-value and call-by-name evaluation. This is a phenomenon which simply does not show up when applicative bisimulation is defined over deterministic [1] nor over nondeterministic [19]  $\lambda$ -calculi. The path towards these results goes through a characterization of bisimilarity by testing which is known from the literature [30]. Noticeably, the latter helps in finding the right place for probabilistic  $\lambda$ -calculi in the coinductive spectrum: the corresponding notion of test is more powerful than plain trace equivalence, but definitely less complex than the infinitary notion of test which characterizes applicative bisimulation in presence of nondeterminism [21].

Further work includes a broader study on (not necessarily coinductive) notions of equivalence for probabilistic  $\lambda$ -calculi. As an example, it would be nice to understand the relations between applicative bisimulation and logical relations (e.g. the ones defined in [14]). Another interesting direction would be the study of notions of *approximate* equivalence for  $\lambda$ -calculi with restricted expressive power. This would be a step forward getting a coinductive characterization of computational indistinguishability, with possibly nice applications for cryptographic protocol verification.

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